Boole's untruth tables

The formal conditions of meaning before the emergence of propositional logic

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Abstract

This paper looks into what can reasonably be regarded as truth-table devices in one of Boole's late manuscripts, as a way of addressing Boole's relation to modern propositional logic. A careful investigation of the divergences between those table devices and our current conception of truth tables offers an opportunity to reassess the singularity of Boole's logical system, especially concerning the relation between its linguistic and mathematical aspects. The paper explores Boole's conception of the compositional structure of symbolic expressions, the genesis of table devices from his method of development into normal forms, and the non-logical origin of the constants 0 and 1 as dual terms. Boole's system of logic is in this way shown to be chiefly concerned with the problem of the formal interpretability conditions of symbolic expressions, rather than with the truth conditions of logical propositions.

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Introduction

The introduction of truth tables in the study of logic bears a special historical significance. Indeed, truth tables have played a decisive role in securing the foundations of the whole logical edifice, providing the means for the first proofs of completeness and consistency for the propositional fragment of logical systems, helping to generalize their properties and perspectives, and contributing

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to establishing a conception of logical thought as chiefly oriented by the formal treatment of the truth conditions of logical propositions.[1]

Associated with that event is the pioneering work of George Boole. Although it is agreed that no actual truth-table device was explicitly drawn by Boole, the essential ideas later associated with the emergence and uses of truth tables are frequently recognized as stemming in one way or another from his work. In the next pages, I shall challenge that received view by suggesting that Boole did indeed present what can be considered as truth-table devices, but that, at the same time, such devices cannot be correctly understood in propositional and truth-theoretical terms, and ask for a different interpretation of the nature of Boole's logical endeavor. I will first examine Boole's place in the existing literature on the history of truth tables, focus next on the different aspects of the singularity of Boole's system with respect to modern propositional logic, and finally thereby explain the original meaning of Boole's table devices.

1 The place of Boole in the history of truth tables

The precise moment of the introduction of truth tables in the history of logic, and the person to be credited for their invention, has been the object of some debate in recent decades. The received history points invariably to the years 1920-22, when explicit truth tables were proposed and used simultaneously by Post Post (1921), Wittgenstein Wittgenstein (2001) and Łukasiewicz Łukasiewicz (1921), who in turn recognized sources of inspiration in Frege, Schröder, Venn, Jevons and Boole. This origin has already been suggested by Quine (Van Orman Quine, 1966, p. 27) and Anscombe (Anscombe, 1963, p. 23), being somewhat endorsed by William and Martha Kneale, who nevertheless make additional efforts to link this emergence to the works of Boole and Frege (Kneale and Kneale, 1971, p. 532).

However, in 1997, Shosky Shosky (1997) discovered some handwritten comments by Wittgenstein on the back of one of Russell's manuscripts, suggesting that both logicians were certainly discussing and working with explicit truth tables as early as 1912. Shosky's article, which also addresses the roles of Frege and Boole as precursors of truth tables, is not only significant for that particular discovery, but also because it proposes a conceptual framework for a rigorous historical study of the question, from which philosophical consequences can be drawn. In particular, Shosky advanced the fruitful distinction between the truth-table *technique* and the truth-table *device*, the former pointing to "the logical process of examining all truth values for a proposition", the latter referring instead to "the mechanical creation of vertical columns of possibilities, measured against horizontal rows of logically exhaustive options" (Shosky, 1997, p. 13).

Following Shosky's positioning of the problem, Anellis Anellis (2004, 2012) provided evidence to support the idea that credit for the introduction of the first truth-table device should be granted to Peirce, who had presented several ways

[1] See, for instance, Gödel's recently published 1939 logic course at Notre Dame Adzic and Dosen (2017), which constitutes a flawless account of the historical, conceptual and technical significance of truth tables for the understanding and development of propositional logic.

of dealing with truth values through matrices in texts and manuscripts, years and even decades before the notes of Russell and Wittgenstein discovered by Shosky.[2] In Anellis and Abeles (2016), Anellis also refers to Dogson (Lewis Carroll) as having worked with incomplete truth tables, following Bartley's account in his edition of Carroll (1986).

Significantly, apart from a few references to premodern logicians, as well as to Frege regarding truth-table techniques, the emergence of a truth-table device appears as deeply rooted in the Boolean tradition of logic. In effect, Jevons, Venn, Carroll, Schröder, Peirce and even Russell[3] were all openly engaged in perfecting and expanding the logical system first introduced by Boole in his seminal treatise *The Mathematical Analysis of Logic* (hereafter MAL) Boole (2009), and later developed in his most celebrated work *The Laws of Thought* (hereafter LT) Boole (1854). Moreover, there are solid conceptual reasons to believe that the decisive development of propositional logic that took place around the 1920s is intimately connected with Boole's pioneering conception of logic. Indeed, truth tables bear an internal link to Boolean algebra, logical propositions rely on a generalization of Boolean connectives such as "and", "or" and "not", and the computable aspects of propositional calculus assured by truth tables can be carried out through a binary arithmetic, composed only of 0 and 1, of which Boole is known to have provided the first rudiments as part of his logical calculus. For those reasons, the historical account of truth tables implicitly or explicitly points to Boole's original system as containing, in some essential sense, the principles underlying truth-table devices. At the same time, the clear absence of truth-table devices in Boole's principal works forces one to think that Boole's own formulations remain in some way insufficient to necessitate the introduction of those devices as such, so that they could only be introduced by his followers, once his system had been conveniently elaborated.

Among the studies concerned with the history of truth tables, that of W. and M. Kneale presents the most explicit argument putting forward the importance of Boole's work for later developments. In their view, Boole's main achievement in this sense is given by his theory of elective functions, which should be understood as a theory of truth functions, only "a short step" away from "Frege's use of truth-tables (i.e. tabulations of alternative truth-possibilities) in his *Begriffsschrift* of 1879" (Kneale and Kneale, 1971, p. 420). Their claim is based on a combination of two aspects of Boole's formulations. The first of them is given by Boole's suggestion in MAL and LT to consider the algebraic—or "elective"—symbols x, y, z... of his system as admitting only the values 1 and 0, which could be interpreted upon certain occasions as meaning that the propositions X, Y, Z... are, respectively, true or false. For the authors, this provides "all that is needed for an interpretation of Boole's system in terms of the truth-values of propositions with the symbols 1 and 0 standing respectively for truth and falsity" (Kneale and Kneale, 1971, p. 413).

However, that circumstance alone is not enough to attest the existence of a truth-table technique in Boole's work. This is certainly the reason why the

[2] See Grattan-Guinness (2004) for Grattan-Guinness's argument in favor of Shosky's position.

[3] Unlike the rest of the names in this list, Russell's relation to the Boolean tradition was not direct, but mediated in a decisive way by Peano's original elaboration, which, in turn, explicitly relies on the formulations of Boole, Schröder and Peirce (see Russell's (Russell, 2010, §2) and Peano's (Peano, 1973, §§vii, xii, xvi, xvii, xxi)).

authors bring up a second aspect in the work of the English logician, namely his method of expressing secondary (i.e. hypothetical) propositions by algebraic symbols, especially in MAL, where Boole presents the following "scheme" (Boole, 2009, p. 50):

	Cases.		Elective expressions.	
1st	X true, Y true	. . . .	xy	
2nd	X true, Y false	. . . .	$x(1-y)$	(1)
3rd	X false, Y true	. . . .	$(1-x)y$	
4th	X false, Y false	. . . .	$(1-x)(1-y)$	

Like the first, that second element alone cannot be thought to contain the fundamental principles of truth tables, since this tabular array only intends to present the way of expressing secondary propositions in the system, and does not provide the means to compute their truth or falsity. In other words, Boole's schema is supposed to work as a dictionary rather than as a truth-table device. For the former to work as the latter, the algebraic or computable properties of the expressions in the last column should be determined in some essential way by a combinatorial treatment of truth values. This is what the first aspect considered by W. and M. Kneale could provide.

As close to each other as those two elements may appear from our contemporary perspective, their connection in Boole's work does not seem to be necessary, to an extent that Boole not only never integrates them in his two major works, but, as the authors correctly acknowledge, he even abandons in LT the direct correspondence between algebraic expressions and the truth values of propositions that seems to motivate the table (1), in favor of an interpretation of symbols such as x in terms of the *time* during which a proposition X is true (Boole, 1854, XI).[4] W. and M. Kneale cannot but see in Boole's withdrawal from a truth-theoretical conception of logic a "return to an unsatisfactory account of truth put forward by some ancient and medieval logicians", blaming metaphysical and theological influences on the logician (Kneale and Kneale, 1971, p. 414).[5] Their guess is that Boole is preparing a logic, not of truth and falsity, but of necessity and impossibility. In this way, the Kneales' interpretation could explain why Boole did not go so far as to develop modern truth tables, although he introduced the basic principles of a modern truth-theoretical understanding of logical propositions that would make it easy for his followers to take that "short step".

Although W. and M. Kneale's reading can illuminate the positive relation between Boole's work and truth tables, the reasons that we can draw from it to explain the absence of truth tables in Boole are rather unconvincing. In contrast, Shosky's analysis provides, in my view, the best existing account of the conditions preventing truth-table devices from actually emerging within Boole's system. While recognizing the principles of a truth-table technique in

[4] For Boole's temporal interpretation of secondary propositions, see Godart-Wendling (2000).

[5] It is hardly surprising, from this point of view, that the connection of secondary propositions with the theory of probabilities, which occupies more than a third of LT, appears to the authors as "obscure" (Kneale and Kneale, 1971, p. 414). For a treatment of Boole's theory of probabilities, see Hailperin (1986); Durand-Richard (2012).

Boole's use of 1 and 0 in LT, Shosky correctly notes that those principles are independent from Boole's tabular arrays such as (1): "Boole did not himself combine the truth-table technique with his use of [an] embryonic truth-table device. [...] it is surprising that he developed both techniques independently..." (Shosky, 1997, p. 14). Yet after acknowledging this circumstance, Shosky affirms that their combination into a unified powerful tool is indeed not necessary in Boole's system, since the existence of truth tables as we now know them is, in fact, related to the definitional conditions of material implication and material equivalence, two components that do not really find a place in Boole's work. Relying on that argument, Shosky concludes his analysis of Boole's place in the history of truth tables suggesting that "[i]f someone could demonstrate Boole's reliance on a truth-table device to develop his logical system, then Boole would have to be given credit for initiating the advance to the new logic with an even greater insight than his discovery of Boolean Algebra or the existential fallacy." (Shosky, 1997, p. 16).

We can leave aside for the moment the fact that the use of truth tables attributed to Frege by the Kneales is more than questionable, and that Shosky's direct association of Boolean algebra with Boole is, in fact, inaccurate.[6] As we will show, the understanding of Boole's use of 0 and 1 as informed by a truth-functional approach or a truth-table technique will also need to be challenged. However, what is certainly more significant when considering Boole's relation to truth tables is that those and other accounts of the possibility and impossibility of truth tables in his work overlook a decisive circumstance, namely that *what can reasonably be considered as truth-table devices are in fact present in the work of Boole*. If they did not appear in the debates around the history of truth tables, it is certainly because those devices can be found not in Boole's main published logical works but in a manuscript belonging to the years that followed the publication of LT. In those pages, entitled "On the Foundations of the Mathematical Theory of Logic and on the Philosophical Interpretation of Its Methods and Processes", the logician goes back over the formulation of LT in order to provide a condensed version of his system in no way lacking in originality compared with previous presentations.[7] Under the "Symbolical expression of the formal laws of logic" section of that manuscript, we can find the following three passages:

p. 92:

> ...the condition $xy = 0$ demands that the values of x and y should be so chosen that their product should vanish. And this restricts the actual selection to the following pairs of values viz.:

1st	$x = 1$	$y = 0$
2nd	$x = 0$	$y = 1$
3rd	$x = 0$	$y = 0$

[6] See the introduction to section 2 below.

[7] Extracts from this manuscript were published in 1952 by Rhees in Rhees (1952), but the editor, manifestly interested above all in Boole's perspective on the problem of reasoning, skipped the section containing the "tables" in question, mentioning only that "[t]he logical symbolism is the same as in the *Laws of Thought*" (Rhees, 1952, p. 238). The publication of the entire document had then to await Grattan-Guinness and Bornet's edition of Boole's manuscripts in 1997 Grattan-Guiness and Bornet (1997), the same year as Shosky's paper.

and excludes the combination $x = 1$ $y = 1$.

p. 92-93:

$$y(1 - x) = 0$$

Logically interpreted this condition demands that there should exist no individuals in the class y which are not found in the class x. [...] Interpreted in the dual Algebra it would permit the combinations

$$\begin{array}{ll} x = 1 & y = 1 \\ x = 1 & y = 0 \\ x = 0 & y = 0 \end{array}$$

and excludes the combination

$$x = 0 \qquad y = 1$$

p. 93:

$$x(1 - y) = 0$$

Logically interpreted this demands the non-existence of the class whose members belong to the class x and not to the class y. [...] Interpreted in dual Algebra it permits the combinations

$$\begin{array}{ll} x = 1 & y = 1 \\ x = 0 & y = 1 \\ x = 0 & y = 0 \end{array}$$

and excludes the combination

$$x = 1 \qquad y = 0$$

From a contemporary point of view, those three passages unequivocally present the truth tables for the propositions we would nowadays write: $\neg(x \wedge y)$, $\neg(\neg x \wedge y)$ and $\neg(x \wedge \neg y)$, corresponding to the expressions $xy = 0$, $y(1 - x) = 0$ and $x(1 - y) = 0$ in Boole's system. If one is permitted to express the combinations "selected" or "permitted" by 1 and those "excluded" by 0, then Boole's tabular arrays could be rearranged without much effort in the following standard truth-table way:

x	y	$\neg(x \wedge y)$	$\neg(\neg x \wedge y)$	$\neg(x \wedge \neg y)$
1	1	0	1	1
1	0	1	1	0
0	1	1	0	1
0	0	1	1	1

The two different aspects of Boole's system that historical accounts identified as motivating its possible relation to truth tables enable an assessment of the relevance of the tables drawn by Boole in his manuscript. Those tables show not only that both dimensions *are not independent* in Boole's system, but that they are actually combined *in his work*. Indeed, the pages in question integrate Boole's regular use of a tabular device[8], with the exhaustive combinations of 0 and 1 by which the truth of a given compound logical expression can be (and is indeed) computed. Boole's remarks concerning those novel tabular arrays leave no room for doubt concerning his conscious intention of integrating these two dimensions of his system:

> It appears then that there exists a perfect formal identity between Logic represented by symbols [...] and the dual Algebra [...]. Upon this identity the methods developed in the Laws of Thought are founded. I have not however in that treatise so fully considered the grounds of the relation upon which its methods rest as I have done in the previous sections of this paper. (Grattan-Guiness and Bornet, 1997, p. 94)

In other words, Boole's expression of logical propositions through elective symbols motivating tabular arrays, and his use of 1 and 0 in the form of a dual algebra allowing computations over those symbols, are formally one and the same thing. What is more, this identity necessitates recourse to a novel tabular device bearing all the elementary properties usually attributed to truth tables, even if they only appear here as embryonic principles in need of further adjustment and systematization. This evidence would suggest that Boole's place in the history of truth tables, and more generally, in initiating the "advance to a new logic", needs to be reassessed, as suggested by Shosky, and the conditions usually associated with the emergence of truth tables would need to be made compatible with those motivating Boole's pioneering arrays.

Significantly, however, the difficulty those tables present resides elsewhere.

2 Boole's singular system

The hitherto unnoticed presence of table devices in Boole's work could certainly reinforce the picture of Boole as the father of modern formal logic. This picture, sketched at the beginning of the previous section, is composed of three main features, each of them defining in a decisive way a specific aspect of modern logical thought. First, Boole would have provided a novel formal *language* to logic, given by a system of logical propositions structured as a Boolean algebra, in which $\wedge$, $\vee$, and $'$ are respectively interpreted as the truth functional logical connectives of conjunction, disjunction and negation. Second, he would have conceived a powerful *calculus* of logic, in the form of a Boolean ring (binary or modulo-2 arithmetic), offering that language the means of deduction over the truth values of its propositions. Third, he would have secured an *ontology* for logic, given by the structure of classes and their relations of intersection, union and complementarity ($\cup$, $\cap$, $'$), as a model of Boolean algebra, assuring the

[8] Which borrows the form from tables such as (1), as we can infer from the first column of the first of the three tables in the manuscript.

semantics of its language (i.e. what that language talks about).[9] Thus, Boole would have proposed, for the first time in history, a formal language, a deductive calculus and an objective semantics as internal aspects of a unified system, whose solidity is guaranteed by the isomorphic relations between Boolean algebras, Boolean rings and their respective models. Thus understood, later logicians only had to amend and refine such a system, following the perspectives initially opened by Boole. In particular, truth tables would progressively affirm their place in that configuration as a formal device assuring the correct articulation between the different components of the system, and especially between the formal language and the deductive calculus.

This general picture is however challenged by the known fact that Boole's own formulations differ in many respects from the system of Boolean logic as we know it today. Indeed, if attention is paid to Boole's original texts, it immediately appears that his own system of expressions does not correspond to a Boolean algebra, his calculus does not follow the rules of a Boolean ring, and the models of his system are far from being standard. This divergence is normally explained as the effect of the understandable hesitations and mistakes of every pioneering endeavor, with natural development by faithful followers progressively eradicating such flaws. The history of that evolution is usually indistinguishable from the history of Boolean logic itself, starting with Jevons's critiques of Boole's system in Jevons (1890), and continuing through its different rearrangements in the works of authors such as Schröder, Peirce and Peano, until Russell's full reassessment of symbolic logic in his *Principia*. This teleological perspective, whose most precise expression concerning Boole's work can be recognized in Corcoran and Wood (1980); Corcoran (2003)[10], is not completely absent in the existing literature on the history of truth tables mentioned in the previous section. This presentist use of history in logic unquestionably contributes to shedding light on the analytical properties of our existing logical systems. However, not only might it prevent the faithful reconstruction of Boole's original thought, but it may also inhibit the capacity of history to suggest courses of development alternative to the ones we know.

Among the efforts to reconstruct the coherence of Boole's original formulations, the work of Hailperin Hailperin (1981, 1986) stands out both for its precision and exhaustiveness.[11] His formalization of Boole's formulations shows that the latter's system neither corresponds to a Boolean algebra (where $1 + 1 = 1$), nor to a Boolean ring (where $1 + 1 = 0$), but to a commutative ring with unit having no additive or multiplicative non-zero nilpotents, as a consequence of Boole's singular additive law, where $1 + 1 = 2$, and more generally, $x + x = 2x$. Moreover, Hailperin identifies the structure of signed multisets (i.e. sets capable of having multiple and even negative occurrences of their elements) as a privileged model for that formal system. Significantly, unlike propositional logic informed by Boolean algebras, Boole's system is thus proved to be undecidable, since the formal system in question contains the same symbols as the theory of integers (Hailperin, 1986, p. 140-141).

Hailperin's work provides invaluable tools for positively assessing the diver-

[9] For a standard presentation of Boolean algebras and rings, see, for example, Givant and Halmos (2009).

[10] Reference to a larger number of works following this approach can be found in Brown (2009).

[11] Other works with the same direction include Brown (2009); Burris (2014).

gences between Boole's own system and Boolean logic, especially regarding the relation between the aforementioned computational and the semantic aspects. However, if one compares Hailperin's formal reconstruction to the canonical forms of Boolean logic, it must be admitted that the relevance of Boole's original system is rather limited. Indeed, Hailperin is forced to acknowledge that Boole never made use of the possibilities of his richer formal structure, mentioning only minor results that the latter could have guaranteed.[12] This is why the presence of what retrospectively appear as embryonic truth-table devices in Boole's manuscripts offers an opportunity to assess the singularity of Boole's system anew, since the divergences they manifest with our current understanding of truth tables direct the attention to the linguistic component of Boole's system, relatively understudied from this original viewpoint.

To this end, we need to address those divergences less as a barrier to overcome and more as a positive difference to understand. Now, if Boole's tables appear to us as embryonic—that is, unborn and perfectible—truth-table devices, it is because we presuppose that Boole's logic is essentially conversant with propositional logic, in which the composition of propositions is determined by the use of connectives, which are in turn governed by truth-functional principles. From this point of view, Boole's tables cannot but show a triple lack of systematicity: in the choice of the propositions whose table is given, in the principles of composition of such propositions which those tables do not capture, and in the relation between the arguments and values of the functions computed by the tables which they somewhat fail to present. In the next three sections, I will address those issues as a way of revealing an alternative systematicity in Boole's system.

2.1 Boole's logic of propositions is not propositional logic: the logical meaning of expressions

The systematic character of modern truth tables is attached to two main different principles of exhaustiveness. On the one hand, truth tables provide a decision procedure to determine the truth or falsity of any proposition, given the truth value of its elementary components, following the hierarchical order of their exhaustive decomposition. For instance, the truth value of the proposition $p \wedge (p \to q) \to q$, given that p is true and q false, can be determined by computing first the truth value of $p \to q$, then that of $p \wedge (p \to q)$, and finally that of the entire proposition, which in this case will turn out to be true.[13] On the other hand, given all the possible truth-value attributions for a given number of different elementary propositions, the exhaustive consideration of the possible different truth tables for the composition or combination of those propositions provides a closed list of all the possible definable connectives ($2^{2^2} = 16$ possibilities for two-valued binary connectives).[14] Both principles of exhaustiveness guaranteed by truth tables contribute in a decisive way to establishing the fundamental properties of the entire system of propositional logic as such. Indeed, if in the first case, one considers all the possible attributions of truth values for the elementary components, truth tables can provide, in addition, a decision procedure determining whether any given proposition is logically true (i.e. true

[12] See, for instance, (Hailperin, 1981, p. 178) and (Hailperin, 1986, p. 147).
[13] See, for instance, (Post, 1921, p. 167).
[14] See, for instance, (Wittgenstein, 2001, § 5.101).

irrespective of the truth of its elementary components, as is the case in the given example).[15] In the second, considering all definable truth tables can, in turn, provide the means to determine the functional completeness of different choices of the respective connectives as primitive terms.[16] Both systematic principles associated with truth tables help to back up the idea that logic can and must be conceived as primarily concerned with the truth conditions of propositions, regarding which truth tables can give the best possible formal definition. Both truth tables and propositional calculus appear then as inseparable from those principles.

Interestingly, none of those principles is, strictly speaking, present in Boole's system, either in his novel use of tables or in his system as a whole. If we want to understand the original meaning of Boole's tables, it is then necessary to call attention to a surprisingly neglected circumstance, namely that Boole's own system is not primarily concerned with propositions as such, but rather with *expressions*. More precisely, Boole's "Calculus of Logic" is above all conceived as a system of symbolic expressions through which logical propositions accept a mathematical treatment. Despite Boole's original perspectives, later developments of his logic would tend to erase the difference between the notions of proposition and expression by reducing the latter to the former in such a way that the notion of expression appears to us as a regular word used to speak about propositions. However, regarded exclusively in terms of the latter, Boole's systems appears cumbrous, clumsy and unnecessarily sophisticated. As a matter of fact, very little of Boole's own system is comprehensible without attributing to the notion of expression a different—yet no less technical—meaning than that of proposition. Not only does Boole never mistake one for the other, but, as we will see, his very conception of logic is consciously built around the problematic connection between them.

In the most general terms, expressions appear as the elements of a pure symbolical dimension in which different domains of objects can be alternately represented. The origin of this notion, as far as Boole's usage is concerned, is mathematical. In effect, the notion of expression inherited by Boole can be traced back to 18th century Continental analysis, marked by the efforts of Euler and Lagrange to define analytical functions as "expressions of calculation".[17] At the turn of the 19th century, the English algebraists, starting with Woodhouse, followed by mathematicians such as Babbage, Peacock and Gregory, were to take up this notion and develop it in the original direction of a purely symbolical approach, that is, of a reduction of the meaning of expressions to the properties resulting from the laws of combination to which those expressions are subject (the notion of expression being thus specified as a combination of individual symbols). As a consequence, symbolical or abstract algebra was established as an independent field of mathematics, based on algebraic operations conceived as pure symbolical manipulations of expressions, regardless of their (numerical or quantitative) content. Within this framework, abstract algebraic laws were considered as admitting as many applications as domains of objects accepting representation by its symbolic expressions, without being burdened with the

[15] For Gödel's use of truth tables for the decidability of propositional calculus, see (Adzic and Dosen, 2017, p. 23).

[16] See (Adzic and Dosen, 2017, § 1.1.8).

[17] For an interesting study putting forward the notion of expression in Lagrange, see Ferraro and Panza (2012).

quantitative import traditionally associated with them.

Among the English algebraists, the originality of Boole was then to realize that those purely abstract manipulations of expressions could also be capable of accounting for the fundamental properties of logical thought. Thus, when simple expressions or symbols (like x or y) accept combination following specific laws (Boole's famous "laws of thought", such as $x(u+v) = xu + xv$, $xy = yx$ or $x = x^2$), such symbols can express the properties of logical concepts. Logical *propositions* can then be expressed as *equations* between those symbols, such as $wy = x - y$, symbolically expressing the conditions stated by those propositions.

By means of this novel interpretation, the algebraic calculus becomes an algebra or calculus of logic, and more significantly, logical thought gains access to an entirely new formal dimension.[18] However, expressions and propositions do not become identical for that reason, and the universe of symbolic expressions, with which Boole's calculus of logic is principally concerned[19], remains essentially larger than that of propositions.

The relevance to our investigation of the distinction between expressions and propositions lies in that only the latter are essentially related to truth and falsehood. On the contrary, the symbolic expressions of Boole's calculus are, as such, indifferent to truth. Certainly, such expressions can happen to be directly or indirectly concerned with the problem of truth, namely when they represent (i.e. accept being interpreted as) propositions. This can take place in two different ways. We have already mentioned the first, when expressions are put in equational form. In this case, their relation to truth is indirect, since symbolic expressions only express the class content of the propositions they represent, and the truth or falsity of the latter does not find a way of being explicitly expressed in the symbols (e.g. the truth value of the proposition "All Xs are Ys" is not actually itself expressed by the equation $xy = x$ that represents it, but only its class content).

The second case occurs when the proposition represented by an equation is hypothetical (or "secondary", as Boole calls it, in contrast with "primary" or categorical), since in this case the symbolic terms that compose the equational expression represent propositions themselves, as in the schema in (1). Under this secondary interpretation, symbolic expressions are more directly concerned with the truth and falsity of the propositions they now somewhat express. As Boole puts it, "Secondary Propositions are those which concern or relate to Propositions considered as true or false" (Boole, 1854, p. 160). It is then this particular interpretation that has motivated the invariable understanding of Boole's entire system in terms of propositional logic. However, some remarks about the limits of that understanding are in order, starting with Boole's aforementioned hesitations regarding the proper way to conceive of this secondary interpretation of expressions involving, among others, the notion of time, and associated with his constant need to attach interpretations to class contents.[20]

[18]This new formal character is here given by the mathematical notion of analytical *form*, related to the formal use of series expansions, rather than by the traditional "*argumenta in forma*" of classical logic. For the formal use of series in 18th and early 19th-century mathematics, see Ferraro (2007, 2008).

[19]Boole is constantly clear about this: "We might justly assign it as the definitive character of a true Calculus, that it is a method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation." (Boole, 2009, p. 4).

[20]See (Boole, 1854, XI, § 5).

Furthermore, Boole's secondary interpretation of his symbols is precisely that: secondary. In other words, the entire system is developed (be it in MAL, in LT or in the manuscripts) first and foremost under the categorical interpretation, and the hypothetical interpretation is only introduced once all the features of the calculus have already been laid out.[21] Boole never stopped insisting on the subordinate place occupied by secondary propositions in his system, leaving the fragment of it directly concerned with truth and falsehood precisely circumscribed.[22]

Yet there is a deeper reason preventing the projection of propositional logic onto Boole's logic of propositions. If one takes a closer look at Boole's formulations, it appears that even under a secondary interpretation, symbolic expressions fail to properly express propositional truth and falsehood as directly attached to logical connectives. Let us refer, for instance, to the treatment of secondary propositions in MAL, where Boole formulations are as close as they can get to modern propositional calculus. In those pages, Boole proposes expressing compound propositions like "X and Y are simultaneously true", "either the Proposition X is true, or the Proposition Y is true", etc., based on a secondary interpretation of $x = 1$ and $x = 0$ respectively as "Proposition X is true" or "Proposition X is false" (Boole, 2009, p. 51). The proposition "X and Y are simultaneously true" will then turn out to be represented by $xy = 1$, as one would expect from a contemporary perspective. However, Boole represents the proposition "X and Y are simultaneously false" either as $(1 - x)(1 - y) = 1$ or as $x + y - xy = 0$. It already appears that from the viewpoint of expressions, both propositions appear essentially different. Certainly, the literal symbols x and y permit one to identify that both are composed of the same elementary propositions, X and Y. However, that is pretty much all that the propositions are shown to have in common.

It follows that no direct correspondence between the form of the expressions and that of propositions necessitates or encourages the analysis of the latter in terms of truth values (true and false) and logical connectives (like "and" or "either . . . or", etc.), since no autonomous, identifiable symbolical principle can be found for them at the level of expressions.[23] In particular, 1 and 0 do not directly represent truth and falsehood, as much as algebraic operations do not directly represent logical connectives. Indeed, in Boole's calculus, the same true compound proposition can be represented by expressions equated either to 1 or to 0, as much as conjunctive propositions can be represented as the multiplication or addition of terms or conjunctive and disjunctive propositions can be represented by the same algebraic operation. In this setting, truth values and logical connectives are *distributed*, as it were, through the entire form of the expression, following a structure radically foreign to that of logical propositions, which Boole never intended to modify.

Accordingly, without autonomous truth values and detachable connectives, a truth functional approach to logical propositions has virtually no role to play

[21] For the reduction of hypotheticals to categoricals, see Nambiar (2000).

[22] As in this passage, in which truth and falsehood are said to concern only "a branch" of logic: "A denial must be a denial of the truth of a proposition and there is a branch of Logic [...] which relates to propositions in their special attributes of truth and falsehood and in the relations flowing from those attributes" (Grattan-Guiness and Bornet, 1997, p. 59).

[23] Boole's modifications in the treatment of secondary propositions in LT do not significantly change this decisive feature of his approach. See (Boole, 1854, p. 169 sqq.).

in his system. The expressions through which Boole's system represents "X and Y are simultaneously true" and "X and Y are simultaneously false" can (and will) accept a functional expression, such as $f(xy)$. However, this functionality is neither a *truth*-functionality (i.e. $f(xy) = 1$ and $f(xy) = 0$ do not necessarily and unequivocally represent that $f(xy)$ is respectively true and false), nor does it permit defining propositional compositionality in terms of logical connectives (i.e. $f(xy)$ and $g(xy)$ are not intended to ultimately represent different logical connectives by which x and y can compose a new proposition).

At the source of this (to our modern eyes uncanny and flawed) way of dealing with logical propositions is the pervading disparity between expressions and propositions upon which Boole's system is built. Such a disparity is incompatible with a conception of logic based upon the truth conditions of propositions. Indeed, Boole's calculus of logic is neither definable nor comprehensible in those terms. As in the case of functionality, it follows that if Boole arrays are indeed table devices, they cannot primarily be *truth*-table devices. Accordingly, the systematicity of the former cannot be measured against that of the latter. If Boole's tables are systematic, their systematicity should be sought elsewhere.

2.2 The method of development and the compositional form of expressions

If we free logical expressions from their traditional relation to truth, and focus again on the elementary expressions upon which Boole's system is based, such as xy, $x + y$ or $x - y$, it will appear that, indifferent to truth and falsehood, expressions are only governed by abstract laws. As such, the main danger they face is not of being false, but of being *meaningless*. In other words, the major problem for a system of symbolic expressions is that they reveal themselves incapable of admitting an interpretation. Here again, Boole inherits this concern not from logic but from the mathematics of his epoch.[24] By preserving the central disparity between symbolic expressions and their propositional interpretation within his system of logic, the English logician forces logical thought to assume the problem of meaning or interpretation as a chief concern. Indeed, unlike the circumscribed character of the problem of truth in his system, Boole's entire logical production clearly shows that his most important philosophical and technical efforts have been directed at the problem of the interpretation of its expressions.

Although the philosophical grounds of his answer to the question of interpretation evolved throughout his work, Boole never abandoned the general idea that the logical interpretation of expressions relies on the capacity of expressions to capture the class content that organizes the logical properties of propositions.[25] However, Boole never actually developed a rigorous theory of classes that could externally control the semantics of his system. Instead, building upon an original and powerful philosophy of signs whose importance can be seen to increase from MAL to the manuscripts after LT, he constantly addressed the problem

[24]Indeed, the meaning of uninterpretable algebraic expressions, such as $\sqrt{-1}$ or non-convergent series expansions, was a common concern for the English algebraists, and occasioned the originality of their approach. For an extensive treatment of this problem in the most general terms, see Babbage's Babbage (1826).

[25]See, for instance, (Boole, 2009, p. 4-5), (Boole, 1854, p. 28, 47), (Grattan-Guiness and Bornet, 1997, p. 67 sqq.).

of interpretation through the formal properties of the expressions themselves.[26] The evolution of this question in his work shows an effort to simplify and unify the principles underlying the interpretability conditions of the system of logical expressions. As we will see, Boole's table devices are the most developed result of that attempt.

From the point of view of a formal system, the problem of meaning or interpretation does not touch the isolated symbols (like x or y) but their combinations into compound expressions, since the formal laws by which that combination is effected is by definition foreign to the meaning of the symbols, and hence likely to introduce meaningless expressive features. If truth tables occupy such a central role in propositional calculus, it is because in them expressions are restricted to propositions, and their relevant logical meaning is reduced to their truth value, in such a way that the problem of their interpretation can be solved by uniquely determining the compositional principle of propositional expressions in terms of truth functions. The combinatorial properties of truth-table devices handle, in turn, those truth functions in an elegant manner. We have already seen how this truth-conditional approach, which truth tables implement, provided a general deterministic procedure to analyze any proposition into its hierarchical components and compute its truth value following the structure of that hierarchy (p. 9).

Indifferent to both propositional forms and truth-conditionality, the expressions of Boole's system cannot afford such a simple solution. Therefore, to find an equivalent device in Boole's logic we should focus less upon their superficial tabular aspect than the central role they play in the articulation between the syntactic structure of propositional expressions and their logical meaning or content.

From this perspective, the key to Boole's system lies not in his table devices but in his *method of development* of logical functions. This method, which Boole considers the process whereby "the analytical element of reasoning finds expression" (Grattan-Guiness and Bornet, 1997, p. 97), contains the essence of Boole's solution to the problem of interpretability. Considered as one of the most obscure components of Boole's logic, the details of and reasons for this procedure are rarely assessed in the literature. This is hardly surprising since it bears no immediate connection to the traditional principles of modern propositional logic; and yet, as aberrant as it may seem, the method of development was one of the most fertile devices in Boole's work. Indeed, Boolean table techniques and devices, as well as Boolean dual uses of 1 and 0, can be archaeologically traced back to this procedure. It is thus worth expounding upon it.

Boole introduced his method of development of logical functions in MAL, under the title "Properties of Elective Functions", (Boole, 2009, p. 60 sqq.) after having presented all the principles of expression (of categoricals, syllogisms and hypotheticals). The aim was to introduce the method by which his system was capable of attaining the generality announced from the very first pages of his treatise. The starting point is, therefore, a functional approach to logical expressions: the logical or "elective" functions $\phi(x)$ and $\phi(xy)$ thus represent any expression involving respectively the elective symbols x or x and y. If

[26] This reliance on signs is what justifies Boole's late definition of logic as "noetic" instead of "ostensive" (Grattan-Guiness and Bornet, 1997, p. 72). For a condensed overview of Boole's late articulation of the problems of interpretability and signs, see Rhees's introduction to Rhees (1952), especially p. 17 sqq.

a functional approach to logic exists in Boole's work, it is here, and not at the level of truth conditions, that it can be found. However, the recourse to functionality is not limited to a notational shorthand. For, surprisingly, Boole then proposes expanding or developing those functions following McLaurin's theorem, in ascending powers of x, based on the fact that elective symbols "combine according to the laws of quantity" (Boole, 2009, p. 60). The resulting development for $\phi(x)$ is:

$$\phi(x) = \phi(0) + \phi'(0)\,x + \frac{\phi''(0)}{1 \cdot 2}x^2 + \&c. \tag{2}$$

Since the developed expression is considered to be symbolically equivalent to what we may term the "enveloped" one $\phi(x)$, it must also be subject to the laws of his system, and in particular, to $x^2 = x$. All the ascending powers of x in (2) can then be substituted by x and then factored out to obtain:

$$\phi(x) = \phi(0) + x\{\phi'(0) + \frac{\phi''(0)}{1 \cdot 2} + \&c.\} \tag{3}$$

Next, the coefficient of x in (3) can be substituted by $\phi(1) - \phi(0)$, since, making $x = 1$ in (3), we obtain

$$\phi(1) = \phi(0) + \{\phi'(0) + \frac{\phi''(0)}{1 \cdot 2} + \&c.\} \tag{4}$$

and subtracting $\phi(0)$ from both sides of (4):

$$\phi(1) - \phi(0) = \phi'(0) + \frac{\phi''(0)}{1 \cdot 2} + \&c. \tag{5}$$

Finally, the aforementioned substitution of (5) in (3) results in

$$\phi(x) = \phi(0) + x\{\phi(1) - \phi(0)\} \tag{6}$$

which Boole rearranges by distributing x and factoring out $\phi(0)$ to obtain

$$\phi(x) = \phi(1)\,x + \phi(0)\,(1 - x) \tag{7}$$

thus getting a recognizable expression of his system. Boole immediately generalizes this result to functions of multiple variables, by developing them with respect to each one of them successively. Thus, for a function $\phi(xy)$ of x and y (such as xy or $x + y$), he obtains:

$$\phi(xy) = \phi(11)\,xy + \phi(10)\,x(1-y) + \phi(01)\,(1-x)y + \phi(00)\,(1-x)(1-y) \tag{8}$$

where $\phi(11)$ means that in $\phi(xy)$ both x and y are to be replaced by 1.

Considered from the perspective of propositional calculus, the eccentricity of Boole's approach cannot be explained away. However, if we bear in mind that the main units of Boole's system are not logical propositions but expressions, Boole's procedure becomes perfectly intelligible. In effect, series expansions (or "developments") were established by the English algebraists as the privileged formal principle for analyzing any symbolic expression into its elementary constituents. Through series expansions, expressions such as $\frac{1}{1-x}$ or e^x could be seen as "composed of", as it were, (an infinite number of) elementary terms of the form $a_i x^i$ (x^n in the first case, $\frac{x^n}{n!}$ in the second), the particular form of which (i.e. the particular coefficients of the different powers of x) is determined as a function of the initial expression. In this way, all the possible symbolical relations between the terms falling under the initial expression can be captured and analyzed as relations between the terms of its expansion.[27] Detached from their numerical conditions, the development of pure symbolic expressions was thus proposed and used by the English algebraists as a fine tool of symbolical manipulation and analysis. In particular, developments provide a common symbolic structure where the multiplicity of symbolic expressions could be projected and measured (e.g. the differences and similarities between $\frac{1}{1-x}$ or e^x can be reduced to those of x^n and $\frac{x^n}{n!}$, which share the same underlying form). In the end, by projecting every symbolic expression onto this elementary shared structure, the general method of development furnishes a powerful instrument of symbolical reduction to *normal form*.

It is then perfectly understandable if, confronted with the problem of the internal compositional form of logical expressions, Boole appeals to the best instrument of symbolical analysis at hand, given by the formal development of functions. Viewed retrospectively, we can think that the insights stemming from the resultant analysis of logical expressions might offset the lack of foundational rigor.

Two major ideas deserve to be mentioned in this sense. The first one is that, by means of this symbolical analysis, any expression subject to the law $x^2 = x$ (i.e. any logical expression) appears to be "composed of" a particular configuration of all the possible combinations of its elementary symbols and their negations. For instance, any expression involving the logical symbols x and y will be analyzable in terms of xy, $x(1-y)$, $(1-x)y$ and $(1-x)(1-y)$ (where $1-x$ can be interpreted as the complementary of x). This is to say that the method of development provides Boole with a *table technique*, in the sense proposed by Shosky. However, it is not a *truth*-table technique, not only because, as we have seen, symbolic expressions are primarily indifferent to truth and falsehood, but also because Boole's table technique, unlike truth-table techniques in the history of logic, is not based on logical grounds (such as the principle of excluded middle), but on the purely formal properties of the symbols subject to the law $x^2 = x$.

From the point of view of the symbolic form of logical expressions, Boole's techniques differ radically from the truth-conditional analysis of propositions. Indeed, a logical expression such as $x + y$ will not be in this case analyzed as composed of x and y, combined through the truth-functional connective $+$, but composed by a certain configuration of the elementary constituents xy, $x(1-y)$, $(1-x)y$ and $(1-x)(1-y)$, as any other expression involving x and y. Interestingly, from this perspective, simple expressions such as x are not necessarily elementary, since they can also be shown to be composed of a certain combination of x and $1-x$ (namely, that combination in which the latter constituent is absent).

Boole's development of logical functions is therefore closer to modern disjunctive normal forms than to truth-table techniques. Yet strictly speaking, the constituents give only the general schema of the normal form. Specific functions are, however, reduced (i.e. developed) to particular cases of that schema. For

[27] For all this, see, for example, Ferraro (2007, 2008).

instance, $x + y$ is developed into $x(1-y) + (1-x)y$ and xy simply into xy. This is where lies the second major logical idea that Boole derives from the symbolical analysis effected by formal series expansions. As a consequence of the operations underlying the method of development, that specific configuration is shown to be the result of alternately attributing the values 1 and 0 to the elementary symbols in the expression in question. The different possible attributions of the values 0 and 1 in the enveloped expression provide then the different coefficients (or "moduli") that determine the particular configuration of the constituents that characterize symbolic expressions. Boole summarizes the whole procedure in the following terms:

> It is evident that if the number of elective symbols is m, the number of the moduli will be 2^m, and that their separate values will be obtained by interchanging in every possible way the values 1 and 0 in the places of the elective symbols of the given function. The several terms of the expansion of which the moduli serve as coefficients, will then be formed by writing for each 1 that recurs under the functional sign, the elective symbol x, &c, which it represents, and for each 0 the corresponding $(1-x)$, &c, and regarding these as factors, the product of which, multiplied by the modulus from which they are obtained, constitutes a term of the expansion. (Boole, 2009, p. 63)

If we take, for instance, $\phi(xy) = xy$, we have that the different possible attributions of the values 1 and 0 to x and y give 1 for $x = y = 1$ and 0 for the others. Taking the values 1, 0, 0 and 0 as the coefficients of the constituents of the development of $\phi(xy)$, it appears that only the first of them is to be kept, namely xy. Employing the contemporary language of normal forms, we can say that in this system the expression xy "reduces" to itself.

A slightly more complex example will help us grasp Boole's method in a more general way. Let $\phi\,(xy) = x\,(1-yx)$. We then have that:

$$\begin{aligned} \phi\,(11) &= 1\,(1-1\cdot 1) = 0 \\ \phi\,(10) &= 1\,(1-0\cdot 1) = 1 \\ \phi\,(01) &= 0\,(1-1\cdot 0) = 0 \\ \phi\,(00) &= 0\,(1-0\cdot 0) = 0 \end{aligned} \tag{9}$$

Hence, substituting the values of (9) in (8), we have:

$$\begin{aligned} x\,(1-yx) &= 0xy + 1x(1-y) + 0(1-x)y + 0(1-x)(1-y) \\ &= x(1-y) \end{aligned} \tag{10}$$

If we consider how these two major ideas are intrinsically connected by the formal use of series expansions, we can understand then that Boole's theory of developments is the very place where the two dimensions of truth tables that were hitherto thought to remain separate actually articulate, namely a table technique and the computation of the content of compound expressions out of all the possible combinations of 1 and 0. The reason why none of these relates directly to truth conditions should be clear by now but their intimate articulation is no less the occasion for the emergence of a table *device*. Indeed, it is almost explicit in the equations we have laid down in (9); and if in MAL Boole gives these results in a line[28], in LT the spatialization of the technique overrides the economy of space, and Boole writes, for the function $1 - x$ expressed as a function of x and y (Boole, 2009, p. 76):

When	$x = 1$	and	$y = 1$	the given	function	$= 0.$
	$x = 1$	,,	$y = 0$	,,	,,	$= 0.$
	$x = 0$	,,	$y = 1$	,,	,,	$= 1.$
	$x = 0$	,,	$y = 0$	,,	,,	$= 1.$

which already shows the fundamental properties of a table device.

Nonetheless, that table is incomplete under the standards of the method of development, since it does not present the constituents, which Boole invariable expresses in equational form, as in (10). We could try to fill this last gap by showing that Boole's method of development contains all the necessary elements of later table devices by projecting them into the embryonic truth-table devices of Russell and Wittgenstein Shosky (1997) or Peirce Anellis (2012). The general schema given by (8) would then be:

$\phi\,(xy)$	y	$(1-y)$
x	$\phi(11)$	$\phi(10)$
$(1-x)$	$\phi(01)$	$\phi(00)$

and the tables corresponding to the two examples $\phi\,(xy) = xy$ and $\phi\,(xy) = x\,(1 - yx)$ are as follows:

xy	y	$(1-y)$
x	1	0
$(1-x)$	0	0

$x\,(1-yx)$	y	$(1-y)$
x	0	1
$(1-x)$	0	0

Yet as a matter of fact there is no need to appeal to such later table devices, since table devices naturally emerge from ordinary algebraic practice if we only stack several developments, such as those in the given examples, in the following way:

$$\begin{aligned} xy &= 1xy + 0x(1-y) + 0(1-x)y + 0(1-x)(1-y) \\ x\,(1-yx) &= 0xy + 1x(1-y) + 0(1-x)y + 0(1-x)(1-y) \end{aligned} \tag{11}$$

If we now remove the superfluous constituents, as is standard in algebra, and place them as headers of the resulting array, we can easily see the table device implied in the underlying algebraic practice of Boole's development:

	xy	$x(1-y)$	$(1-x)y$	$(1-x)(1-y)$
xy	1	0	0	0
$x(1-yx)$	0	1	0	0

(12)

From here, we only have to transpose the resulting table to arrive at a modern (truth) table:

	xy	$x\,(1-yx)$
xy	1	0
$x(1-y)$	0	1
$(1-x)y$	0	0
$(1-x)(1-y)$	0	0

[28] For example: "Let $\phi(xy) = x(1-y)$, then $\phi(10) = 1$, $\phi(11) = 0$, $\phi(01) = 0$, $\phi(00) = 0$" (Boole, 2009, p. 74).

2.3 Brief archaeology of 1 and 0. Dual algebra as foundations of the expressive system

Along with an original analysis of logical expressions into normal form, Boole's method of development brought forward the relevance of the joint functioning of 1 and 0 in his system. Considering the principles of expression set out in MAL, that feature is rather unexpected and Boole must have become aware of its importance retrospectively, since only in the postscript to MAL (written while the work was already in print) does he acknowledge that 0 and 1 are the "points" in which "[t]he two systems of elective symbols and of quantity osculate, if I may use the expression". He improvises here an explanation in terms of the principle, which he will not later reproduce, that "a Proposition is either true or false" (Boole, 2009, p. 82). Notwithstanding this fairly marginal justification *a posteriori*, the meaning of 0 and 1 as dual terms in Boole's system resists any substantial association with truth and falsehood, and remains deeply rooted in the most delicate symbolical mechanisms of the mathematics of his time.

To corroborate this point, we have only to consider the development of expressions such as $x + y$ or $\frac{x-y}{y}$, and stack them together with our previous examples in (12). We obtain then:

$$
\begin{array}{r|cccc}
 & xy & x(1-y) & (1-x)y & (1-x)(1-y) \\
xy & 1 & 0 & 0 & 0 \\
x(1-yx) & 0 & 1 & 0 & 0 \\
x+y & 2 & 1 & 1 & 0 \\
\dfrac{x-y}{y} & 0 & \dfrac{1}{0} & -1 & \dfrac{0}{0}
\end{array}
\qquad (13)
$$

The new expressions are perfectly legitimate within Boole's system, and by no means artificial; yet they bring an unexpected behavior into Boole's table devices, and into his system as a whole, since they cause the introduction of numerical terms (and in some cases, such as $\frac{1}{0}$ and $\frac{0}{0}$, not even clearly numerical) other than 0 and 1. As trivial as these examples may be, they make apparent what we already knew: that Boole's table techniques and devices are not *truth*-table techniques and devices, and that Boole's functional approach to logic cannot be a *truth*-functional approach. For what could 2, −1, and even $\frac{1}{0}$ and $\frac{0}{0}$, mean as legitimate values in a truth table? And which truth value could we attach to them as values of truth functions? Boole's logical functions might very well range over a domain consisting only of 0 and 1, their codomain remains open to the self-governed realm of symbolic expressions, and the problem of attaching to them a reassuring logical interpretation remains an open issue.

Yet what then could be the logical significance of 0 and 1 as dual terms? Notwithstanding the received wisdom, there is no unified concept subsuming the varied uses Boole makes of 0 and 1 throughout his work. What we find instead is a multiplicity of locally more-or-less effective interpretations which the logician tries, with much effort, to progressively unify, without necessarily succeeding. In more than one sense, it could be claimed that if the problem of the general interpretability of the expressions of his system animated Boole's entire logical enterprise, the elaboration of a sound and unified theory of 0 and 1 relentlessly oriented his search for a solution.

Previous to its dual use in the method of development in MAL, the symbol 1 had been introduced independently from the very first lines of the work, to represent "*the Universe*", as comprising "every conceivable class of objects whether actually existing or not…" (Boole, 2009, p. 15).[29] The most immediate logical use that Boole makes of it as an elective symbol is in combination with the operation of subtraction to express negation: as already suggested, $1-x$ expresses the class of elements that are not x. Significantly, in MAL, 0 is not equally introduced as an elective symbol. In fact, 0 is not introduced at all in his early logical work. It appears for the first time when expressing logical propositions, such as "no Xs are Ys" in the form $xy = 0$ (Boole, 2009, p. 21). At this early stage of Boole's formulations, 0 belongs to the system only as the result of the admitted properties of algebraic operations over symbolic expressions. Under this particular use, 0 is not, like 1, a standalone symbol. Its meaning, if any, results from its inscription in the context of an equational expression, in which case Boole interprets it roughly as *non-existence* (i.e. there exist no individuals of the class X that are at the same time individuals of the class Y). Note that none of these primitive uses of 1 and 0 is immediately or fundamentally related to the notion of truth. Furthermore, under such asymmetric interpretations, 1 and 0 do not even function as an identifiable pair. Truth and falsehood, in the context of secondary propositions, will provide the first occasion upon which they are conceived as constituting an alternative, following the equational use previously attached to 0. We have nevertheless already shown the limits of such an interpretation for a standalone conception of 0 and 1.

It follows that the first neat dual use of 0 and 1 in Boole's work is effected by the method of development in MAL. Resulting from the algebraic properties of expressions, Boole shows no intention of assigning any logical meaning to them in that work, other than the hasty remark in the postscript. However, the relevance of this novel association between both terms will gain a central role in the reconfiguration of the system operated in LT.

Boole's 1854 work manifests an evolution in the philosophical grounds of his logical calculus, from operations over classes to linguistic categories supposed to express them (nouns, adjectives, verbs...). In this new context, 0 and 1 have no immediate place among the symbols of the system (in particular, 1 cannot be introduced as a primitive symbol representing "the Universe"). Boole will then introduce them, together this time, in a surprising way: 0 and 1 are the only "symbols of Number" subject to the fundamental law of logic $x^2 = x$ (Boole, 1854, p. 32). In other words, considered as regular algebraic expressions, $x^2 = x$ accepts only two roots or solutions, which are precisely 0 and 1.[30] Boole concludes that logical symbols can therefore be compared to quantitative

[29] This definition is, as we now know, full of subtlety and danger. However, Boole adroitly avoids defining 1 as the *class* of all classes, and prevents the confusion between both by attributing to them different *semiological* status: in Boole's own terms, while 1 is a "symbol" (as much as x or y), classes are referred to by "letters" such as X or Y. See (Boole, 2009, p. 15).

[30] Boole will further associate the fact that the fundamental law of logic is of second degree to the dual nature of logical symbols (since from $x^2 = x$ we can have $x(1 - x) = 0$, which he can interpret as the law of contradiction). Incidentally, this offers him occasion to consider the possibility of the fundamental law being of higher degree, thus explicitly envisaging the existence of logics whose terms would accept more than two possibilities (Boole, 1854, p. 49-50). However, Boole never explicitly relates the degree and factorization of that fundamental equation to the duality of 0 and 1, which could have been made through their status as roots of that equation.

symbols admitting only the values 0 and 1, and considers the existence of a specific algebraic domain to that effect:

> Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. (Boole, 1854, p. 37)

It is then clear that under that primitive understanding as dual terms, 0 and 1 are first and foremost *numbers*, without any logical meaning. As such, they do not actually belong to the calculus of logic, but constitute a parallel system to which the algebra of logic can be compared, as a sort of "model" by reference to which analogies can become correspondences. Only after having introduced them as non-logical symbols will Boole try to determine their "logical value and significance" as symbols of his system. It should come as no surprise that the logical value Boole attributes to them is not truth value, but the meanings of "Nothing" for 0 and "Universe" for 1, based on what is possibly one of the earliest purely algebraic definitions of 0 and 1, respectively as absorbing and neutral multiplicative elements (Boole, 1854, pp. 47-48).

The introduction of 0 and 1 as the fundamental terms of a new algebra, which Boole will later call "dual Algebra" or "arithmetic of 0 and 1" (Grattan-Guiness and Bornet, 1997, pp. 91, 113), provides a much more harmonious and unified presentation of his system. In particular, the duality of both terms having been introduced independently, the presentation of the method of development is hugely simplified. Indeed, by considering symbolic functions $f(x)$ as ranging over the domain of dual algebra (i.e. admitting only the values 1 and 0), Boole introduces its development directly by assuming its "reduction" to the form $ax + b(1 - x)$. It then suffices to consider that $f(1) = a$ and $f(0) = b$ to determine the coefficients and obtain the full development:

$$f(x) = f(1)x + f(0)(1 - x) \tag{14}$$

A generalization to multiple variables is straightforward, as in MAL, and the ancient derivation of (14) by Taylor series expansions is reproduced in a note as being less general than the new one, which "strictly holds, in the logical system, the place of the expansion of $f(x)$ in ascending powers of x in the system of ordinary algebra." (Boole, 1854, p. 72 n.).

One could be tempted to think that with dual algebra, Boole is progressively disposing of all the foreign mathematical background in favor of a purely formal approach, and therefore coming closer to a modern conception of Boolean algebra that will unavoidably orientate his logical system in the direction of propositional logic. However, the truth is that dual algebra does not alter the fundamental problem with which Boole's logic is concerned. On the contrary, it confirms it and clarifies it. It confirms it, because the same non-dual coefficients (i.e. other than 0 and 1) continue to haunt the development of logical expressions, reasserting that despite their derived logical meaning, 0 and 1 do not lose their primitive numerical nature. This corroborates the idea that *the principal task of a calculus of logic is to deal with the boundary that distinguishes logical expressions from all the other expressions that necessarily surround them.*

Yet dual algebra also provides novel means by which to clarify that problem, as Boole's tables reveal in his manuscripts, the complex sense of which we are finally ready to understand.

3 Boole's untruth tables. The formal conditions of expressive meaning

During the entire previous section we saw that to Boole the general problem of interpretability constitutes a key concern of formal logic. By defining the form of its expressions in terms of the truth conditions of logical propositions and by reducing the meaning of the latter to their truth values, propositional calculus prevents this problem from appearing. However, Boole wants the form of the symbolic expressions of his system to be independent from that of propositions, and determined only by abstract laws. The analysis of their compositional form is then not enough to trivially solve the problem of their logical meaning. As we have seen, the method of development by which such an analysis takes place constantly produces uninterpretable results. A general method of interpretation complementing that of development is needed, so that every symbolic expression representing a logical proposition finds a proper interpretation. Referring to the procedure by which he actually solves this major problem, Boole resumes the situation in the following terms: "though *functions* do not necessarily become interpretable upon development, yet *equations* are always reducible by this process to interpretable forms." (Boole, 1854, p. 78. Boole's emphasis).

Boole's solution thus aims to guarantee that any equational form is interpretable. In MAL, Boole obtains this result by building on certain properties of functions. Thus, in the "Properties of Elective Functions" chapter (Boole, 2009, pp. 60-69) he shows in turn that functions differ only by the moduli of their development (i.e. developments constitute a sound representation of functions); that each of the constituents of the development satisfies the law $x^2 = x$ (i.e. is interpretable in his system); that the product of any two of them is equal to 0 (i.e. they are pairwise mutually exclusive); and that their sum is equal to 1 (i.e. the decomposition partitions the universe). Based on those results, he then proves that for any equation of the form $V = 0$ (V being an elective function) the constituents of the development of V with non-zero coefficients should be separately equated to 0 [p. 64-65], and more generally, that for any equation $w = V$ (w being an elective symbol not contained in V), any constituent of the development of V affected by a coefficient other than 0 or 1 can be separately equated to 0.[31]

With all this, Boole is finally ready to provide the required general method of interpretation. Given an equation, we can solve for one of its symbols in terms of the others to obtain an equation of the above form $w = V$, and then develop the second term. All the constituents having 1 as coefficients are to be kept; all those with a coefficient 0 should be discarded. If non-dual coefficients appear, either they are of the form $\frac{0}{0}$, in which case the value is "indeterminate" (borrowing from the arithmetical meaning of this expression) and hence we should replace it

[31] Interestingly, Boole remarks that the first of those two results can lead to $1 = 0$, which he interprets as the "nonexistence of the logical Universe", and indicate the attempt "to unite contradictory Propositions in a single equation" (Boole, 2009, p. 65).

with an arbitrary elective symbol v. Any other numerical coefficient (including $\frac{1}{0}$) indicates that the corresponding constituent should be separately equated to 0, this latter condition expressing a "denial of existence" of the corresponding class (Boole, 2009, p. 77).

To briefly illustrate Boole's method, take for instance the equation $wy + y = x$. Solving for w, we obtain $w = \frac{x-y}{y}$, whose second term we have already developed in (13). Applying Boole's method of interpretation, we obtain in the end the system:

$$\begin{aligned} w &= v(1-x)(1-y) \\ x(1-y) &= 0 \\ (1-x)y &= 0 \end{aligned}$$

which is fully interpretable in logical terms as meaning that w is composed of an indefinite part v of elements that are neither x nor y and that, furthermore, elements that are x and not y as well as elements that are y and not x do not exist under the conditions expressed by the initial equation.

Boole's solution at this stage is by all means inelegant and uneasily sophisticated. As with the method of development, Boole is here drawing from algebraic analysis rather freely[32] and without much regard to the overall consistency of his system. The introduction of dual algebra in LT offered an opportunity to reflect in terms of a new systematicity but apart from some unconvincing efforts[33] the essential components of the interpretation procedure, as well as its formal grounds, ultimately remained unchanged.

Manifestly unsatisfied with this circumstance, Boole had occasion to return to the question of interpretability in the 1856 manuscript which contains the table devices that motivate the present pages.[34] In the first pages of this text, Boole refers to "the imperfect mode in which [the Mathematical Theory of Logic] has been presented as a philosophical system in the Laws of Thought." (Grattan-Guiness and Bornet, 1997, p. 64). Once his new account was exhibited based on a fresh use of dual Algebra, he expounded on the reasons for that imperfection:

> The identity of the formal laws of operation [of Logic represented by Symbols and of dual Algebra] was demonstrated [in LT] but not the fact that the formal conditions of interpretability are the same also and that these conditions are a necessary consequence of the formal laws. (Grattan-Guiness and Bornet, 1997, p. 94)

Boole is here referring directly to the new approach his table devices present. It is not about the truth conditions of propositions, then, but about the *formal*

[32] Boole explicitly associates the whole procedure with that of solving linear differential equations, "arbitrary elective symbols in the one, occupying the place of arbitrary constants in the other" (Boole, 2009, p.70).

[33] For instance, Boole tried to confer a logical signification on the uncomfortable symbols $\frac{0}{0}$ and $\frac{1}{0}$, much like in the case of 0 and 1, by associating them with indefiniteness and infinity (Boole, 1854, pp. 90–91). In the manuscripts he will also refer—no less unconvincingly—to the four coefficients in terms of "logical categories", now considering $\frac{1}{0}$ to be the symbol of impossibility (Grattan-Guiness and Bornet, 1997, pp. 99–100).

[34] The text, entitled "On the Foundations of the Mathematical Theory of Logic and on the Philosophical Interpretation of Its Methods and Processes" (Grattan-Guiness and Bornet, 1997, pp. 63–104), provides a complete overview of his system, insisting on its foundational aspects. It was intended as a preliminary philosophical introduction to the system for the purpose of an application of the theory of probabilities.

conditions of interpretability of expressions, determined by dual algebra as a non-logical domain. Boole's idea is relatively simple. It consists in extending the fundamental law of logic $x^2 = x$ from individual symbols to any expression, that is, to any combination resulting from the composition of those symbols. Thus, if V represents "any combination of symbols", the formal condition of logical interpretability (i.e., the formal condition for such an expression to be a logical expression) is given by the form $VV = V$. Unlike previous uses of this form in his work[35], Boole's strategy in this manuscript is to derive from it particular conditions of interpretability for every possible law of composition of logical symbols at their syntactical level (i.e. addition, multiplication, etc.). The idea looks so simple that one might wonder why Boole did not come up with it before. The answer is certainly to be found in the absence of division from the primitive operations in the previous versions of his calculus. Despite that absence, the presence of division in symbolic expressions is pervasive as a consequence of applying the inverse of multiplication in the symbolical manipulations. The method of interpretation was thus forced to deal with it and with its regrettable effects for the coefficients of the expansion (i.e. $\frac{0}{0}$ and $\frac{1}{0}$), without being able to tackle the problem at its root.[36] The 1856 manuscript, in contrast, introduces division at the same primitive level as the formal operations of addition, subtraction and multiplication (Grattan-Guiness and Bornet, 1997, p. 79 sqq.). Expressions do not become more interpretable for that reason, but the treatment of interpretability in general can now benefit from a new systematicity.

Thus, given the logical symbols x and y (satisfying the law $x^2 = x$), there are altogether four and only four ways in which expressions can be combined to form new expressions: xy, $x + y$, $x - y$, $\frac{x}{y}$. Apart from multiplication, which is unconditionally interpretable, all of them are subject to interpretability conditions, since they can lead to uninterpretable expressions. In the previous pages of the manuscript, Boole had already derived those conditions conceptually, so to speak, as properties of the operations of conception they are supposed to express (composition, addition, subtraction and abstraction). The general condition $VV = V$ henceforth also allows for their formal derivation.

Let us take, for instance, subtraction, as a way of combining the logical symbols x and y to obtain the expression $x - y$. Then, following the general condition $VV = V$, such an expression will be logically interpretable if and only if

$$(x - y)(x - y) = x - y$$

which can, in turn be transformed by symbolical manipulations in the following

[35] The presence of this condition can be traced back to MAL, recognizable under the form $\{\phi(xy\ldots)\}^n = \phi(xy\ldots)$, which Boole presents as a condition introducing "symmetry into our Calculus" (Boole, 2009, p. 66). In LT, Boole acknowledges it, under the form $V(1 - V) = 0$, as "the condition of interpretability of logical functions", and devotes a whole chapter to exploring some of its properties (Boole, 1854, p. 93 and ch. X). In those pages, such a condition inspired, however, a rather convoluted method for reducing every step of a calculation to an interpretable form.

[36] For the problem of division in Boole, see Hailperin (1986).

form:

$$x - xy - yx + y = x - y$$
$$y - xy + y - yx = 0$$
$$y(1-x) + y(1-x) = 0$$

This results in the last expression only being able to be satisfied if $y(1-x) = 0$. Thus, $y(1-x) = 0$ appears as the ultimate condition of logical interpretability for expressions that combine logically interpretable symbols by means of subtraction. Setting aside multiplication, a derivation analogous to the one just given[37] yields the following respective conditions of interpretability for the remaining three compound expressions: $xy = 0$, $y(1-x) = 0$, $x(1-y) = 0$.

Although in equational form, those three expressions do not primarily represent logical propositions. They are particular formal conditions corresponding to the compositional laws of expressions by which a logical proposition might be represented. One could think of them as expressions of second order or degree, in the sense that they refer to other expressions and control the dynamics of their syntax with respect to their (propositional) meaning.

Strictly speaking, such expressions only render explicit the conditions of that control. It is their evaluation at the values of dual algebra that will finally furnish the actual conditions of satisfaction allowing us to draw the line between interpretable and non-interpretable cases of the initial compound expressions. Thus, in the case of subtraction, the condition expressed by the equation $y(1-x) = 0$ is satisfied only for all the possible combinations of 1 and 0 except when $x = 0$ and $y = 1$, meaning that any expression involving subtraction will require, in such a case, going through non-interpretable expressions in the process of deduction, since in $x - y$ will then give the value -1 "which is not included in [the dual] system" (Grattan-Guiness and Bornet, 1997, p. 93). The complete formal determination of those conditions of interpretability by dual algebra is, of course, what Boole's table devices provide.

Let us illustrate Boole's new method with the example $y - x + \frac{z}{y}$. Following Boole's tables, the subtraction $y - x$ excludes the combination $x = 1$ and $y = 0$; the quotient $\frac{z}{y}$ excludes $y = 0$ and $z = 1$; and the addition of those two terms excludes $y - x = 1$ and $\frac{z}{y} = 1$. If from all possible combinations of dual values for x, y and z, we exclude those that satisfy such conditions, we have that the following combinations are selected:

$x = 1$	$y = 1$	$z = 1$
$x = 1$	$y = 1$	$z = 0$
$x = 0$	$y = 1$	$z = 0$
$x = 0$	$y = 0$	$z = 0$

and these others are excluded:

$x = 1$	$y = 0$	$z = 1$
$x = 1$	$y = 0$	$z = 0$
$x = 0$	$y = 1$	$z = 1$
$x = 0$	$y = 0$	$z = 1$

[37] In the paragraphs preceding his presentation of the tables, Boole explicitly provides the derivation of condition for the additive expression (Grattan-Guiness and Bornet, 1997, p. 92).

It follows that when used to evaluate the initial expression $y - x + \frac{z}{y}$, the selected combinations yield only interpretable coefficients: 1, 0, 1 and $\frac{0}{0}$, while the others produce non-interpretable coefficients: $\frac{1}{0}, \frac{0}{0}$, 2, $\frac{1}{0}$. One will notice that the coefficient $\frac{0}{0}$ (which Boole continues to consider as interpretable, having a value of either 0 or 1) is present in both groups. Yet while in the first case all the intermediate steps by which this coefficient is obtained are interpretable, in the second this is not the case (we have, namely, that $y - x = -1$).

We can finally positively assess the singular systematicity of Boole's table devices. If we recall the triple lack of systematicity they presented in the anachronistic light of propositional calculus, we can now see that Boole's tables are perfectly systematic in the choice of the expressions they represent, in the compositional analysis they guarantee, and in the kind of functionality they implement. However, such a fragile systematicity is only revealed once we understand that Boole's conception of a formal logical system is less determined by the truth conditions of logical propositions than by the interpretability conditions of expressions.

From this alternative perspective, the expressions $xy = 0$, $y(1 - x) = 0$, $x(1 - y) = 0$ appear then as exhaustively covering the conditions of interpretability of Boole's symbolic expressions by which logical propositions could be represented. Furthermore, the conditions their respective tables express involve a deep conception of the compositional structure of expressions, not borrowed from two-valued logical propositions or any other semantic domain, but resulting from a complex relation between expressions only. Indeed, Boole's new approach supplements the compositional form of expressions given by their development, with a second principle based on the simple syntactical composition. As a result, the internal structure of expressions is ultimately determined by the connection between a principle of decomposition into normal form and a principle of composition through symbolical operations, whose relation to meaning is controlled by other expressions of a necessarily different order (e.g. the compositional structure of the expression $x - y$ is determined both by its developed normal form and the subtractive combination of x and y, and the interpretability at the crossroads of both is controlled by the expression $y(1 - x) = 0$). Finally, if unlike truth-table devices, Boole's tables show a difference between the domain and the codomain of the functions they translate, we can now see that such a discrepancy constitutes a positive instrument for drawing the line between interpretable and non-interpretable expressions (e.g. as "not included in [the dual] system", -1 becomes the positive mark of the non-interpretability of $x - y$ when $x = 0$ and $y = 1$).

Concluding remarks

One could object that there is strictly no merit in looking for something where it is known not to be and being proud of not finding it. However, it is not exactly in the familiar sense that truth tables are absent from Boole's work. If we follow the usual characterization of truth tables in the history of logic (both as a technique and as a device), it can be reasonably claimed that such tables are indeed there. However, although this circumstance may be of some interest given the existing literature on the history of truth tables, what is more surprising is that, whilst Boole's table devices possess everything required to be considered in such a way, they cannot be understood as *truth*-table devices. Their study thus becomes relevant in assessing those disregarded dimensions of Boole's own work and restoring with them lost insights associated with the emergence of the mathematization of logic.

I have shown that, considered from the perspective of their internal necessity within his entire work, Boole's table devices present an image of formal logic that does not necessarily coincide with that stemming from the modern propositional calculus invariably attributed to the English logician. In particular, Boole's conception of a calculus of logic is not primarily concerned with the truth conditions of logical propositions, but with the meaning conditions of symbolic expressions. My attempt to understand the multiple systematic dimensions attached to this problem has revealed the originality of Boole's singular system, in particular with respect to the articulation between a formal language and a deductive calculus. To this end, I have reconstructed his complex understanding of the compositional structure of symbolic expressions, examined the genesis of tabular devices associated with their computational properties, and uncovered a conception of the dual terms 0 and 1 as symbolic devices not yet captured by the logical significations of truth and falsehood. Finally, I have shown how each of those aspects of Boole's thought concurred in proposing a novel device which only on the surface might look familiar to our modern eyes.

The multiple hesitations, mistakes and inconsistencies of Boole's formulation can be taken as read. The lack of rigor does not, however, exhaust what a work such as Boole's has to offer. Thus in closing I cannot but endorse Boole's words in his immediate assessment of his new method (Grattan-Guiness and Bornet, 1997, p. 94):

> I apprehend that the identity of the systems [of Logic and Dual Algebra] as respects not only the formal laws of operation but also the formal conditions of interpretability is a fact of great moment and significance and that it would be unphilosophical to regard it as a merely accidental coincidence.

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