

Frege’s *Habilitationsschrift*: Magnitude, Number and the Problems of Computability

Juan Luis Gastaldi

SPHERE, UMR 7219, Univ. Paris Diderot – CNRS

Abstract. The present paper proposes a new perspective on the place of Frege’s work in the history of computability theory, by calling attention to his 1874 *Habilitationsschrift*. It shows the prominent role played by functional iteration in Frege’s early efforts to provide a general concept of numerical magnitude, attached to an embryonic recursion schema and the use of functions as expressive means. Moreover, a connection is suggested between the iteration theory used and developed by Frege in his treatise and Schröder’s original concern for the mathematical treatment of the problem of the feasibility of algorithmic computation.

Keywords: Frege · Habilitationsschrift · function · iteration · recursion · quantity · magnitude · history · philosophy · Schröder

1 Introduction

The place of Gottlob Frege’s work in the history and philosophy of logic has been profusely acknowledged. Its place in the history and philosophy of theoretical computer science constitutes, however, a more sensitive issue. The usual historical accounts of the theory of computing, even the most recent ones like [6], [11], [35], typically acknowledge Frege as the inventor of the first fully formalized language, in search of more rigorous foundations for arithmetic. As such, Frege’s work is presented as immediately associated to those of Cantor, Peano or Russell. But for the same reason, its significance tends to remain tied to the tragic fate of set theory and the logicist program. From this point of view, the developments around the notion of “effectively calculable number theoretic function” that would take place in the 1930s, leading to the formulation of Church’s thesis as a fundamental landmark in computability theory, seem to arise as a reaction against—rather than as an elaboration of—the logical and philosophical perspectives of a work such as Frege’s, made possible by Hilbert’s formalist attempts to overcome the logicist pitfalls. It should not be surprising then if the fundamental problems defining the region of a computability theory, can hardly be recognized within the classical repertoire of Frege’s concerns. As a symptom of this situation, we can see that more detailed studies in the history and philosophy of computability, like [30], [20], [40], [41], [1] or [5], are more inclined to find a precursor in Dedekind, or even in H. Grassmann or Babbage, than in Frege. The aim of the present paper is then to supplement those historical and philosophical inquiries by calling attention to Frege’s habilitation thesis, a mathematical

treatise in which a certain number of problems associated with computability appear to lie at the basis of the motivation for developing a formalized logical language for arithmetic.

Frege’s *Habilitationsschrift* [14], significantly entitled “Methods of Calculation based on an Extension of the Concept of Magnitude”¹, was presented in 1874 as a thesis to obtain the qualification as *Privatdozent* at the University of Jena. It belongs to the mathematical period of Frege, preceding his logical work, which would only begin five years later, with the publication of the *Begriffsschrift*. Unfortunately, this early period of Frege’s work remains largely understudied among Frege scholars. In the last two decades, works like [44], [46], [36], [19], [42], [43], have in part remedied this situation, without according, nevertheless, much attention to the *Habilitationsschrift*. All things considered, only two studies concerning specifically this text deserve to be mentioned here: Wilson’s postscript [45] to a reprint of [44] and Gronau’s papers [25], [26]. Yet, the specific relevance of this mathematical treatise for the genesis of the problem leading Frege from mathematics to logic still remains to be studied.

The absence of any proper research into Frege’s habilitation thesis is all the more surprising once we notice that, despite its weighty mathematical content—but also attached to it—the explicit aim of the *Habilitationsschrift* is to provide a non-intuitive account of numerical magnitudes or quantities (*Größe*). In other terms: the same problem that motivated the undertaking of the *Begriffsschrift*, as Frege presents it in the very first pages of his famous booklet. The importance of the *Habilitationsschrift* must then be sought in the conceptual construction, emerging within a purely mathematical framework, of the problem of a non-intuitive account of arithmetic, necessitating an entirely new approach to language that will turn out to change radically the nature of logical thought. Interestingly enough, it is at the level of this construction that one can recognize the presence and the action of a number of principles that half a century later would prove to be fundamental for the emergence of a rigorous concept of the informal notion of effective calculability.

2 Mathematics as of 1874

A detailed account of the mathematical context of this work is not possible within the limits of our contribution. It should nevertheless be noticed that none of the seminal texts of the logicist tradition is available at the time Frege composes his habilitation thesis. That means that the developments which would contribute to the shaping of the logicist program, to which Frege is invariably associated, cannot be rigorously considered as the context of this work. Unfortunately, Frege

¹ In German: “*Rechnungsmethoden, die sich auf eine Erweiterung des Grössenbegriffes gründen*” [17], reprinted in [18]. The German term *Größe* is usually translated both as “magnitude” and “quantity”. In the following, we will use both terms interchangeably, preferring the latter when speaking specifically about Frege’s text, in order to agree with the published English translation, which also accepts the adjectival form “quantitative”.

gives no explicit references that could help establish its sources in this case. Yet, the introductory remarks of his treatise point unambiguously to the *pure theory of magnitudes* (*reine Grössenlehre*) which, under the inspiration of Gauss' work, had been conducting the process of arithmetization of analysis since the beginning of the 19th century.²

Nevertheless, this general context needs to be doubly relativized. First, Frege's adherence to arithmetization "in the Berlin Way" (that is: in the style of Weierstrass) is highly questionable.³ His relation to the Gaussian background of the theory of magnitudes must then be rather conceived in terms of what came to be known as "the conceptual approach", that is, the intention to build mathematical theories out of a conceptual widening inspired by the internal relations between the objects under consideration.⁴ As it will appear from the analysis of the *Habilitationsschrift's* introductory remarks, this is the sense Frege gives to his "extension of the concept of magnitude". Second, Frege's habilitation thesis is contemporary of a multiple effort to detach numbers from magnitudes.⁵ To this trend belongs in particular Dedekind's 1872 brochure on irrational numbers [7], followed by his work on the seminal booklet *Was sind und was sollen die Zahlen?* [8], even if the latter would not be published until 1888. Although independent from those works, Frege's *Habilitationsschrift* participates in the same tendency to derive a rigorous concept of number out of a general notion of magnitude.

Under this general framework, a certain number of mathematical theories and methods belonging to the context of 19th-century mathematics can be thought to converge in Frege's treatise and contribute in one way or another to his elaboration, if only as available knowledge at the time when it takes place. We can thus mention at least three of those fields, starting with the theory of complex functions, and specifically, of *functional equations*, in line with Cauchy's new foundations for analysis⁶. Also, the *symbolic approach to algebraic operations*, first developed by the English algebraists and incipient in Germany through the works of Hankel [28], R. Grassmann [24] and Schröder [39]. We can evoke as well a basic method for introducing natural numbers and defining the elementary arithmetical operations through *recursive definitions*, thanks to the work of H. Grassmann [23] and its later adoption by Schröder [39].⁷

² Gauss' conception of magnitudes is explicitly presented in his short manuscript "Zur Metaphysik der Mathematik" [21], written around 1800. Such a viewpoint can already be found in Kant's philosophy (see, for instance, [29, A713ff, esp. A717]; I owe this reference to Wilfried Sieg). For an analysis of Gauss' conception of magnitudes and its influence on the program of arithmetization, see [12], [2], [34].

³ See, for example, [43]. I borrow the expression "arithmetization in the Berlin Way" from Petri and Schappacher [34, pp. 351 ff].

⁴ For the "conceptual approach", its Gaussian roots, and the difference between this approach and "the Berlin Way", see [12, p. 241 ff], [2, p. 321 ff] and [43, p. 166 ff].

⁵ See the section "The End of the Theory of Magnitudes in 1872" in [34].

⁶ See for example [9], [10].

⁷ Gronau [25] also pertinently evokes the "interpolation problem". For a more comprehensive and detailed account of the general context of Frege's mathematical work, see the above referred [45], [44], [46], [36], [19], [42], [43].

The almost unnoticed birth of what would later become a singular mathematical theory deserves however a closer attention. Indeed, in 1870 and 1871 respectively, Ernst Schröder published two long papers which are considered as the starting point of *iteration theory*. As we will see, this work bears some essential and surprising connections with Frege's *Habilitationschrift*, and it is highly probable that Frege knew Schröder's work at this time. The first of them, called "On infinitely many algorithms for the solution of equations" [37], deals with the problem of a general algorithmic approach to the determination of the (algebraic or transcendental) root of a function $f(z)$, where z is a complex argument of the form $z = x + iy$, conceived as a point in the complex plane. The question of an algorithmic solution is specified by Schröder as that of finding a function $F(z)$ which always returns a value z' closer to the root z_1 than any argument z initially taken within a neighborhood of z_1 . The root z_1 appears as the limit of the sequence $(z^{(r)})$ of these r successive values, or, expressed in terms of F , as the limit of $F^r(z)$, when r approaches infinity, $F^r(z)$ being the r^{th} iteration of F on the initial value z .

The second paper [38], "On iterative functions", addresses specifically the problem of the practical impossibility of computing such algorithms. Schröder begins by laying down his notation for iterative functions in a much neater way than his previous text. An iterative function is now explicitly defined by induction ("in recurrenter Weise"), through the functional equations:

$$F^1(z) = F(z), \quad F^r(z) = F^{(r-1)}\{F(z)\},$$

and the second and third iterations of F on z are respectively written:

$$F^2(z) = FF(z) \quad F^3 = FFF(z).$$

Once this notational apparatus has been laid down, Schröder is ready to introduce the central problem of his paper:

One can now be guided on the *problem of iterative functions* by the needs of computational analysis [*rechnenden Analysis*]. Since even the lightest computations are unworkable [*unausführbar*] if their number is large, one will notice at first sight that, for a tolerably large number r , it is not expedient to determine in this way the final result: $(z)^r = F^r(z)$. [38, p. 297]

Through notions such as "*unausführbar*" (unworkable, unfeasible) or "*practicabel*" [38, p. 297] (practicable, feasible), Schröder not only openly tackles the problem of the computational effectiveness and the complexity of algorithms, but, what is more, calls for a *mathematical* solution. More precisely, Schröder suggests that the execution of the $(r-1)$ substitutions should be avoided and defines the task of making the number of arithmetical operations to be carried out independent of r . This does not mean that the final result given by the quantity $z^{(r)}$ must itself be independent of r , but that an expression of $z^{(r)}$ is to be found in which " r enters [...]" only as a general number, so that it is converted from

an index into an argument, and $(z)^r$ is *given explicitly* as an analytical function of r " [38, p. 297]. Mathematically, this amounts to finding a function $\Phi(r, z)$ of two complex arguments r and z , increasing in the plane r for every point of the plane z , and satisfying the functional equation:

$$\Phi(r, z) = \Phi(r - 1, F(z)), \quad (1)$$

together with the initial condition: $\Phi(1, z) = F(z)$.

3 Frege's Elaboration of the Problem of Magnitudes

Complex functional equations, symbolic treatment of algebraic operations, recursive definition of arithmetical operations and iteration theory—all of those regions of 19th-century mathematical landscape can be identified in Frege's *Habilitationsschrift*, even if their sources are kept silent. However, put in perspective, the significance of this treatise does not so much lie in the development of those fields⁸, than in the way in which Frege articulates them in a comprehensive conceptual construction guided by the problem of providing a notion of magnitude or quantity completely general, and yet capable of as many applications as possible.

3.1 Philosophical Insight: The Non-intuitive Nature of Quantities

The problem that opens the *Habilitationsschrift*, in accordance with the stakes of a conceptual approach to a pure theory of magnitudes, is that of the difficulties raised by complex numbers, even under their geometrical representation, with regard to the classical concept of magnitude or quantity as determined by Euclidean geometry. Already inspired by Gauss' approach, Frege had dealt with the representation of complex numbers in his recent doctoral dissertation, developing an entire geometry of the imaginary, in the hope that a projective extension of Euclidean geometry could help retrieve the intuitive representation complex numbers lacked. The result turned out to be rather convoluted and unfruitful, as Frege himself admits [13, p. 55]. It is hardly surprising then that Frege addresses this time the question of geometrical intuition of quantities altogether, rejecting from the beginning any essential intuitive dimension of complex numbers. However, this does not make complex numbers less "quantitative" than natural ones, inasmuch as other means of determining their quantitative character have been found. And yet, complex numbers do not suppose a modification of the essence of quantity either, since their non-intuitive nature does nothing more than reveal that the intuition of quantity has always been merely apparent. Indeed, as Frege argues: "Bounded straight lines and planes enclosed by curves can certainly be intuited, but what is quantitative about them, what is common to lengths and surfaces, escapes our intuition." [14, p. 56]

⁸ Although the *Habilitationsschrift* does contain some remarkable discoveries in iteration theory, as shown in [25], [26] and [27].

From this follows a clear-cut distinction between arithmetic—as the science of magnitude or quantity—and geometry, based on the grounds of their respective fundamental principles. If intuition lies at the heart of geometry, Frege advances that the construction of arithmetic should be otherwise carried out by the incorporation of the essential propositions of this science into a concept of quantity that “we create ourselves” [14, p. 57]. Intuition in arithmetic is not completely banished though, but relegated to the moment of the mere application, which permits to judge only about the fruitfulness of speculation. The whole problem is thus outlined as that of proposing a concept of quantity embracing the totality of the propositions of arithmetic, and broad enough to allow as many applications as possible.

3.2 Conceptual Construction: from Addition as Fundamental Principle to Functions as Expressive Means

Frege’s initial idea to undertake this task is that all the arithmetical propositions a concept of quantity should embrace refer in one way or another to *addition*, “for the other methods of calculation arise from this one” [14, p. 57]. If this thought, already present in Gauss’ [21], might be more directly inspired by the recursive method of definition of H. Grassmann and Schröder, it can also be seen as arising from Frege’s previous geometrical attempts, as suggested at the beginning of his treatise, where he affirms that all that remains when the Euclidean intuition is lost are “certain general properties of addition” acting between the point of origin and the end point of geometrical figures [14, p. 56].⁹ But by subsuming under addition all those transformations, the *Habilitationsschrift* already engages in the path of a purely conceptual foundation of arithmetic. For it cannot be as a specific arithmetical operation that addition is called for here, but as a general principle underlying all operations as such. That is why Frege advances a characterization of addition as a process, which is rather a new subsumption of addition under abstract operations, than a definition of addition itself: “In the most general terms, the process of addition is as follows: we replace a group of things by a single one of the same species.” [14, p. 57].

Regarded in perspective, one could think that the *Begriffsschrift* was specially conceived to give an adequate scriptural form to those kind of definitions. And indeed, the question of identity of conceptual contents that will organize Frege’s 1879 booklet arises already at this point in which, even if devoid of any mathematical effectiveness, the general characterization of addition has for Frege the virtue of providing a general criterion for “quantitative identity” capable of defining a concept of quantity both general and endowed with “a real content”. Moreover, Frege not only maintains that this conceptual setting encompasses the whole “content of arithmetic”, but explicitly states that natural numbers, as quantities of a special kind, “can also be defined from this standpoint”, even though such a task exceeds the limits of his treatise [14, p. 57].

⁹ See also [13, p. 19].

There is no doubt then that the *Habilitationsschrift* makes us witness the genesis of the problem that will preoccupy Frege for the rest of his work. But the *Habilitationsschrift* is not the *Begriffsschrift*, and even less the *Grundlagen* or the *Grundgesetze*. Precipitating connections with Frege's most revered works could end up obliterating the richness of this early text, which resides in the specific means by which Frege's nascent problem finds an original way to be structured, before merging into that of forging a new script and a new logic. That's why Frege abandons abruptly such embryonic foundational issues and continues its conceptual construction at the level of *operations*, under which the process of addition has been discreetly subsumed. Indeed, Frege introduces abstract operations as perfectly fitting the general characterization for addition just given: "If we repeat an operation f by constantly resubmitting its result to it, we can regard the repeated applications of operation f as new operations" [14, p. 58]. Thanks to this adequacy, quantity can now be attributed to operations—and through them, to the most different processes as well, such as displacements or rotations. In particular, as Frege points out, specific arithmetical operations (like multiplication and exponentiation) can appear, from this point of view, as mere repetitions of other basic operations (like addition), and the same holds for any computation of approximate values, thanks to the application of "recursive formulas" (*Reursionsformel*) [14, p. 58].

Significantly, the position of the problem at the level of repeated operations imposes an organization over the general field of quantity in different regions or domains, depending on the particular initial operation on which the repetition acts (displacements, rotations, multiplication, etc.). Frege calls these regions "quantitative domains" (*Grössengebiete*), and their existence is part of the general definition of quantity; even more, they are what gives its "real content" to the concept of quantity [14, p. 57].

To conclude the conceptual construction of his problem, Frege states the horizon that has thus been drawn: "This should permit us to recognize those parts of arithmetic that would be covered by a theory of the concept of quantity as it relates to functions." [14, p. 58]. If one considers the concept of number as taking over from that of magnitude or quantity, one could hardly find a better expression to qualify Frege's lifelong undertaking. Indeed, there's a sense in which Frege's most general program—which does not necessarily coincide with that of logicism—can be considered settled from this point on, provided that we properly notice the shift that Frege subtly performs in his statement, from the multiple domains of operations to the homogeneous territory of *functions*. Although slight, this shift is not without significance since, as mentioned in the previous section, a symbolic treatment of abstract operations was already an established discipline of 19th-century mathematics. What is more, it was in the region of this theory that a system of mathematical logic emerged through works such as those of Boole, Jevons or Schröder. That is why, by moving from operations to functions, Frege not only gives to the problem of quantity a whole set of new resources, but opens the possibility of an original connection between its conceptual stakes and the means of the theory of functions.

Frege introduces functions abruptly as naturally taking over operations, without any comments. The shift is nonetheless unambiguous: from this point on, the whole theoretical construction of the *Habilitationsschrift* will take place at the level of functions, operations appearing only as specific cases and applications. But even though he does not address the question of the nature of their relation, the way in which this articulation will take place in the rest of the treatise suggests that Frege conceives functions as a canonical mathematical *expression* of operations. In other terms, functions appear as the general form in which operations (like displacements, additions, approximations, etc) are *written* in order to become subject of mathematical treatment. As such, they provide a unified medium for the expression of heterogeneous domains of operations. If a unique notion of quantity is to be constructed for the multiple quantitative domains defined by operations, it is at the unified level of functions that this construction is to be carried out.

3.3 Mathematical Specification: Functional Iteration

Once this whole conceptual framework has been set up, from addition to functions through operations, Frege continues his elaboration of the general problem of quantity by determining its mathematical conditions. The first step is to give a general representation of quantities by means of functional expressions:

After what has been said above it will be understood that we assign to the functions $\varphi(\varphi(x))$, $\varphi(\varphi(\varphi(x)))$ double or triple the quantity of the function $\varphi(x)$. It is no less clear that the function $\psi(x)$ is to be assigned a fourth of the quantity of $\varphi(x)$ when $\varphi(x)$ is identical with $\psi(\psi(\psi(\psi(x))))$, that the quantity $\chi(x)$ is the reciprocal of the quantity of $\varphi(x)$ when $\varphi\chi(x) = x$, and finally, that when x is a function of itself, the quantity of the function must be designated as the null quantity. [14, p. 59]

Immediately deriving from the conceptual framework previously laid out, this singular representation of numerical quantities in terms of *functional iteration* anticipates by more than half a century Church's insights.¹⁰ But at the time of the *Habilitationsschrift*, Frege cannot count on any of the advancements of those golden fifty years of logic, and above all, on the fundamental idea of using functions as logical terms.¹¹ Frege's concept of "quantity of a function" cannot but appear then as a very strange entity in the context of 19th-century complex analysis, which tended to conceive quantities either as the value of a function's argument, or as the value the function assumes for this argument. For, as Frege

¹⁰ Church introduced his numerals in his 1933 paper [4]. In 1922, Wittgenstein had already defined numbers as the "exponent of an operation", written functionally [47, §6.02].

¹¹ This idea will not be introduced until 1879... by Frege himself. An embryonic use of functions in logic can nevertheless be found as early as 1874 in Hermann Lotze [31], one of Frege's sources of influence.

hastens to point out, the quantity of a function is not to be confused with either of them, but must rather be conceived as something that can be identified and extracted, as it were, from the form of the function as such, namely its implicit iterative structure, if any.¹²

A simple example should help illustrate Frege's novel idea. Starting from operations, we can consider, for instance, multiplication as a specific arithmetical process. From a symbolical point of view, this operation—typically written “.” or “ \times ”—can be symbolized by a generic character, like “ \circ ” in “ $2 \circ 3$ ” or “ $a \circ b$ ”. Instead, Frege proposes to express it functionally, *i.e.* in the form of $f(x)$, as in the following cases:

$$\varphi(x) = 2 \cdot x, \quad \psi(x) = 4 \cdot x, \quad \chi(x) = 8 \cdot x . \quad (2)$$

Given all these functions expressing the operation of multiplication, Frege's idea is that a notion of quantity could be built thereupon if we succeed in singling out an iterative structure relating them. This amounts to show that, for example, $\chi(x) = \varphi(\varphi(\varphi(x)))$, in which case the function $\chi(x)$ will be assigned the triple of the function $\varphi(x)$, which will, in turn, be assigned a third of the function $\chi(x)$. In our example, we can easily see that this is the case, since we have:

$$\chi(x) = 8 \cdot x = 2 \cdot (2 \cdot (2 \cdot x)) = \varphi(\varphi(\varphi(x))) . \quad (3)$$

Likewise, $\psi(x)$ is to be considered the double of $\varphi(x)$ (the latter being a half of the former), and $\psi(x)$ and $\chi(x)$ appear respectively as two thirds and three halves relatively to each other.

It can be noticed that this singular functional approach to quantity involves a new way in which functions can be thought of and used. Detached both from the quantities of their arguments and of the values corresponding to them, functions become an instrument to analyze internal relations between given mathematical expressions (terms of arithmetical propositions, in this case). Indeed, it is at the level of their functional expression that $8 \cdot x$ is shown to be quantitatively related to $2 \cdot x$ in (3). From this point of view, Frege seems to rejoin the English abstract algebra, and more generally, the tradition of symbolical thought. Except for the fact that the latter would rather tend to cast numerical terms aside and symbolize all those expressions identically as “ $a \cdot x$ ”. Frege's functional expressions, on the contrary, allows to capture their relation, structuring their difference instead of erasing it, as in $\chi(x) = \varphi(\varphi(\varphi(x)))$. Hence, numerical quantities become functionally expressed or “schematized” rather than symbolically abstracted.

Given that, although used as expressive means, they are not turned into abstract symbols, functions don't lose their mathematical properties (such as continuity or differentiability, among others). As a result of this double nature of functions, the concept of quantity built upon them will not be primarily

¹² Frege seems to be well aware of the strange novelty of this conception since he immediately feels the need to propose a “geometrical picture” providing “an intuitive understanding of the matter” [14, p. 59], even though the *raison d'être* of his new concept of quantity is to stand on non-intuitive grounds.

determined by general laws, like commutativity or distributivity, as in the case of abstract algebra, but by an intimate relation with calculation. If we come back to our example, we can see that if it is possible for us to assign the respective quantities to the three functions $\varphi(x)$, $\psi(x)$ and $\chi(x)$, it is only because, thanks to our knowledge of elementary arithmetic, we are capable of recognizing without much effort that $4 = 2 \cdot 2 = 2^2$ and $8 = 2 \cdot 2 \cdot 2 = 2^3$. Only under this condition are we able to grasp an iterative relation between the initial expressions, informing the functional schema out of which numerical quantities can be attributed to functions. As elementary as this procedure may be, it relies all the same on the mechanisms of actual calculation, which cannot be bypassed without removing the real ground of Frege's entire construction. Conversely, the concept of quantity thus defined opens up to the possibility of furnishing in return new methods of calculation (as stated in the very title of the *Habilitationsschrift*), to be explored as specific solutions to this problem and giving way to a number of concrete applications.

This back and forth movement between a general concept of quantity and actual calculation is so decisive in the *Habilitationsschrift* that the final specification of the problem leading to the solution will be derived from the difficulties associated to this relation. For if we are capable of easily recognizing an iterative structure out of (2), it is of course because those expressions were specially chosen to fit our knowledge of elementary arithmetic. But what if we were confronted to *any* set of expressions, even to those of which we ignore whether they define a common (quantitative) domain or not? Hence, the concept of numerical quantity constructed as a problem at the level of "expressions of calculation"¹³ requires, when addressed in its mathematical generality, to tackle two specific questions, that Frege states with extreme clarity right before stepping into its mathematical resolution:

What is the function whose quantity stands in a given relation to the quantity of a given function?

Do the quantities of two given functions belong to the same quantitative domain, and if so, in what relation do they stand? [14, pp. 59-60]

4 The Solution: An Embryonic Recursion Schema

Writing the general form of the functions under consideration as $X(x)$, we can see that the task of finding an iterative structure between them amounts to that of finding a way to express this general form not only as a function of x but of n as well, *i.e.* $X(n, x)$, where n is the number of the iterations of the given function $X(x)$. Taking a last look at our example, we can see that in the case of multiplication, if the general form of the expressions in (2) is $X(x) = bx$, then the transformed function is $X(n, x) = a^n x$, as we can easily check by writing:

$$\chi(x) = \varphi(\varphi(\varphi(x))) = 2(2(2x)) = 2^3 x = \varphi(3, x) .$$

¹³ Frege will employ the term *Rechnungsausdruck* in his two famous papers dealing with the notion of function [15], [16].

The question is then: what could be the form of this transformation *in general*? The solution requires the establishment of a general form or schema capturing the extraction of the quantity n out of any given function. Here is how Frege introduces it [14, p. 60]:

The answer to these questions presupposes a knowledge of the general form of a function which is n times a given one. In more definite terms, we need a function of n and x which turns into the given function when $n = 1$ and which satisfies the functional equation

$$f(n_0, f(n_1, x)) = f(n_0 + n_1, x) \tag{4}$$

Surprisingly, the general form (4) can be seen as a reformulation of Schröder's equation (1), with the respective initial condition expressed in words.¹⁴ The reason of this connection lies in that both cases have to do with a transformation between indexes, as marks of the repetition of operations, and values, as arguments of functions. In this sense the *Habilitationsschrift* can be thought of as the point of confluence of two main trends: the operational tradition of the English symbolical algebra, and the functional tradition of the Continental complex analysis. As we have seen, those two traditions had already crossed their paths almost imperceptibly in [38] when, guided by the needs of actual computation, Schröder was forced to think of a way to pass from $F^r(x)$ to $\Phi(r, x)$, operating the conversion of r "from an index into an argument". But the significance of this conversion, if not unnoticed by Schröder, remained at least without any real consequences for his later conception of quantity. Frege, on the contrary, identifies in this subtle conversion the exact mechanism out of which a general concept of number can be constructed without abandoning the ground of calculation.

From this new perspective, the functional equation (4), together with its initial condition, constitute nothing less than an embryonic *recursion schema* defining the most general form of Frege's novel concept of numerical quantity.¹⁵

¹⁴ Recalling the fact that x considered as a function is to be assigned the null quantity by definition, then one could also add the condition: $f(0, x) = x$, also implied by Schröder in [37, p. 319].

¹⁵ It might be interesting to compare Frege's form both with Dedekind's famous 1888 "definition by induction" in [8], and with the recursion schema introduced by Gödel's 1931 paper [22]. Making the necessary modifications of function and variable names to facilitate the comparison, we have, for $n_1 = 1$ in (4):

Frege 1874	$\varphi(1, x) = \psi(x)$	$\varphi(k + 1, x) = \varphi(k, \varphi(1, x))$
Dedekind 1888	$\varphi(1) = \omega$	$\varphi(k') = \mu(\varphi(k))$
Gödel 1931	$\varphi(0, \vec{x}) = \psi(\vec{x})$	$\varphi(k + 1, \vec{x}) = \mu(k, \varphi(k, \vec{x}), \vec{x})$

where Dedekind's k' means the number following k [8, §73] (corresponding to $k + 1$ in the case of addition [8, §135]). It appears that Frege's form defines a nested simple recursion, which was shown by Péter [33, §10.3] to be reducible to primitive recursion. Furthermore, if iteration is a special case of primitive recursion, the latter can be in turn reduced to the former as well (see [32, §I.5.10]).

Interestingly enough, Frege’s schema is not conceived in the form of the immediate successor $n + 1$ of a number n , as in all classical foundational approaches to arithmetic—and in Dedekind’s in particular—but as a sum of two non necessarily succeeding numbers n_0 and n_1 . This is all the more surprising since Schröder’s equation (1) was indeed written in terms of successive iterations ($r - 1$ and r).¹⁶ The reason must lie in the conception of number that his construction inherits from his generalized notion of addition,¹⁷ which does not determine numerical quantities progressively in succession, but distributes all the numerical quantities within a certain domain at once. That’s why this schema specifies the nature of quantitative domains as well: a quantitative domain is a family of functions determined by a given function and satisfying the equation (4). Additional analysis of (4) will enable Frege to further characterize functions belonging to the same quantitative domain as related to each other in the form of the logarithm of their first derivatives at their common fixed points [14, p. 60-61].¹⁸

Because Frege’s principal aim is to use iteration to define a concept of numerical quantity, once the general functional equation (4) is laid out, he follows a different path than Schröder, and immediately expresses n as a function ψ of the value of the argument x of the function, and the value X of the function at that argument, defining what he calls “*quantitative equations*”:

$$n = \psi(X, x) . \quad (5)$$

where $X = f(x)$, *i.e.* the given function. Expressed in those new terms, the task amounts to finding functions ψ capable of extracting the quantity n out of functions defining or belonging to a quantitative domain. The condition expressed by (4) is then reformulated in terms of (5):

$$\psi(X, x_0) + \psi(x_0, x_1) = \psi(X, x_1) . \quad (6)$$

To solve those quantitative equations, Frege introduces two methods. The first one is called the *method of substitution*. It consists essentially in constructing new quantitative domains out of known ones, based on the fact that if $\psi(X, x)$ is a solution of (6), then the function

$$\psi(\vartheta(X), \vartheta(x)) \quad (7)$$

is a solution as well. This method confirms the fundamental place given by Frege to the operation of addition, since the function ψ corresponding to it, namely

$$\psi(X, x) = X - x , \quad (8)$$

¹⁶ This difference is also remarkable from the viewpoint of iteration theory, because the functional equation (4) is no other than the so-called “translation equation”, which had not been explicitly given before him (see [26]).

¹⁷ This conception also recalls Gauss’ idea that magnitudes cannot be the object of mathematical investigation in isolation, but only in relation to one another [21, §2].

¹⁸ Frege only occasionally acknowledges the restrictions associated to conditions such as differentiability, variable elimination, etc., which might be a consequence of his interest in the general conceptual construction rather than in rigor. For an account of some of this neglected restrictions, see [25].

is a general solution of (6).¹⁹ For this reason, Frege calls addition the “simplest” of operations [14, p. 59], and uses (8) as the basis schema for the construction of other quantitative domains in a process where is revealed the recursive capacities of (4). Indeed, multiplication is constructed from addition, and exponentiation from multiplication, following the schema (7) and iterating the function $\vartheta(x) = \frac{\lg(x)}{\lg(a)}$. Other quantitative domains are also constructed in the same way by means of other functions ϑ .

Frege attributes “unlimited possibilities” to this method of substitution. However, he admits that the function $\vartheta(x)$ corresponding to a certain desired goal might be difficult to find. That is why he introduces his second method: the *method of integration*. The main idea is to find a function φ relating $X = f(x)$ to $\vartheta(x)$. To do so, Frege considers the possibility of dealing with the quantity n as an infinitesimal. Drawing from the form $X_\delta = x + \delta\varphi(x)$ of a function of an infinitely small quantity δ , he derives—for $X_n = f(n, x)$ when $n = \delta$ —the form:

$$X_\delta = x + \delta \left(\frac{\partial f(n, x)}{\partial n} \right)_{n=0}$$

(because $f(0, x) = x$). He can then define a function φ such that:²⁰

$$\varphi(f(n, x)) = \frac{\partial f(n, x)}{\partial n} . \quad (9)$$

If in (9) we consider x as constant and we solve for dn to recover the quantity n by integration, the result is:

$$n = \int \frac{dX}{\varphi(X)} + C . \quad (10)$$

The right hand term of (10) can then be seen as a function ϑ used in the method of substitution, namely $\vartheta(X) + C$, and the constant C is determined by the fact that n is the null quantity when x and X are equal. Thus Frege obtains:

$$n = \vartheta(X) - \vartheta(x) ,$$

which is the form of the substitution in the functional equation of addition (8). But if Frege retrieves in this way the same form he has introduced in the method of substitution, this whole process of integration has allowed him to determine the function $\varphi(x)$, which is at the same time connected to $X = f(x)$ through (9) and to $\vartheta(x)$ through (10).

What follows is a series of methods to represent φ , including the consideration of fixed points of the given function $f(x)$, the introduction of a functional

¹⁹ On these quantitative equations—nowadays known as Sincov’s functional equations—and their solutions, see [27].

²⁰ The function φ is nowadays called as the “infinitesimal generator”, and (9) is known as the second “Jabotinsky equation” (the first and the third will be introduced in the following pages). See [25].

equation to test if a function is the right φ , a representation of φ as a Taylor series, and a recursive definition of the coefficients φ_k of these series.

Frege's *Habilitationsschrift* continues with a generalization of these results for functions of several variables, and finishes with a series of applications, including an application of those methods to continued fractions, which constitute a way to represent irrational numbers.²¹

5 Concluding Remarks

Far from being inconsequential, Frege's *Habilitationsschrift* confronts us with the emergence of what can be considered to be the fundamental problem informing his long-lasting endeavor, namely that of a purely conceptual construction of arithmetic. If the elementary logical scaffolding needed to achieve this goal will have to wait until the preparation of the *Begriffsschrift*, this 1874 treatise performs a first step in the direction of a conceptual determination of number from a general notion of magnitude, providing a valuable insight on the mathematical context and internal conditions motivating the process of arithmetization.

From this point of view, the *Habilitationsschrift* can be thought of as sharing the same intention as a work such as Dedekind's, namely that of proposing an alternative way to arithmetization, based on a "conceptual approach" to a pure theory of magnitudes that would eventually replace the general notion of magnitude with a precise concept of number. Interestingly enough, recursive mechanisms plays a central role in both attempts. However the singularity of Frege's *Habilitationsschrift* with respect to Dedekind's work—as well as to the classical pioneer works on arithmetization and logicism—lies on its complete independence from a set-theoretic foundational approach to the problem of a conceptual elaboration of number. This circumstance, which could usually be understood as a shortcoming, constitutes in fact what gives Frege's early work all its relevance *vis-à-vis* the history of computability. For the absence of a foundational perspective in the *Habilitationsschrift* is compensated and even outweighed by a concern with the conditions of actual calculation. Indeed, as we have seen, the quantities conceptually defined in Frege's treatise are not only symbolized, as in other symbolic treatments of the "science of quantity", but also thought to be computed out of the values of given functions. Incidentally, the conceptual construction of the notion of quantity is, in turn, intended to furnish new methods of calculation, along with the existing ones.

Needless to say that the notion of calculation is certainly not to be taken here in the sense of the 1930s notion of "effective calculability", if only because the given functions Frege relies on are not, in general, computable in this sense. However, as far as it may be from effective calculability, Frege's concern with numerical calculation gives rise to a number of principles in his conceptual construction which do bear a significant affinity with those informing, more than half a century later, the developments around the question of an "effectively

²¹ For more details in some aspects of this last sections of the *Habilitationsschrift*, see [25], [26] and [45].

calculable number theoretic function". Starting, as we have seen, with the fact that Frege's main mathematical tools are associated to (if not completely derived from) a mathematical treatment of *feasibility problems*, attached to an *algorithmic approach* to root computation. More deeply, it appeared that not only Frege proposes an original notion of numerical quantity constructed upon the usual functions of analysis, through the recognition of an iterative structure at the level of functions themselves, but he also determines such a notion in terms of *functional iteration*. This original approach to numerical quantities, at the crossroads of the symbolical and the functional traditions, involves, moreover, a novel use of *functions as expressive means*, since mathematical functions are handled as instruments to analyze and express internal relations between given mathematical expressions.²² Last but not least, if an *embryonic recursion schema* can be identified in Frege's thesis, it is only through a functional equation established as the expression of a general form of quantity, and manifested by this double nature of functions in the *Habilitationsschrift*—i.e. as numerical expressions to be computed and as expressive means. The *generalized notion of addition* informing this recursion schema, and conceived as the simplest operation from which the other methods of calculation arise, also suggests that the *Habilitationsschrift's* attitude towards calculation is not indifferent to the problems inspiring the emergence of a theory of computability in the 20th century.

It is well known that the articulation of Church's Thesis, as a fundamental landmark in computability theory, was chiefly determined by the developments on recursive functions, tracing back to Dedekind's "Recursion Theorem", evolving through the works of Skolem and Hilbert, and attaining Herbrand and Gödel's definition of the class "general recursive functions".²³ At the same time, the introduction of Church's purely functional language for logic in [3], as well as his iterative representation of positive integers in [4], constituted decisive steps converging to the same goal. Unlike Dedekind's work (and specially unlike his 1888 famous booklet), no direct influence in this history of computability theory could be claimed for Frege's early formulations: the *Habilitationsschrift* remained utterly unnoticed throughout this whole period. However, the significant presence of a number of those ideas in this early work provides a reason to reassess the place of Frege in the transition from a theory of magnitude to a theory of number, independently from the stakes of set theory and the logicist program, which could contribute to a better understanding of the internal conditions of the development of a conception of computability, beyond the limits of historical influences.

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²² Ultimately, the *Begriffsschrift* could be understood as the comprehensive development of this idea, namely as the construction of a language "modeled upon that of arithmetic", organized around functions as expressions of conceptual content.

²³ For a detailed account of this evolution, see [1], [30], [40], [41].

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